EFG based stability analysis of piezoelectric FGM plates subjected to electricity, heat and non-uniformly distributed loads

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Abstract

This paper presents the buckling analysis of the piezoelectric functionally graded material (FGM) rectangular plates subjected to non-uniformly distributed loads, heat and voltage based on the mesh-free method. A two-step solution procedure is implemented. The first step is to determine the pre-buckling stresses of the plates subjected to non-uniformly distributed loads. The second step is to solve the buckling loads and buckling temperatures. The variational form of the system for the calculation of pre-buckling stresses is based on a two-dimensional (2D) plane stress problem, and the variational form with penalty method of the plates for the calculation of buckling loads and buckling temperatures is based on the Mindlin plate assumption. The displacement is approximated using the moving least squares (MLS) technique based on a set of scattered nodes. Two numerical examples are presented to validate the proposed mesh-free method. Keywords: functionally graded plate, moving least squares, buckling analysis, non-uniformly distributed load, heat, voltage.

1 Introduction

Functionally graded materials (FGMs) are a mixture of two or more materials in such way that their material volume fractions vary continuously along certain
dimension. Due to the continuous variation of material properties from one surface to the other, the FGM plate avoids the interface problem that exists in homogeneous composite structures. The FGM can be often seen as a thermal barrier plate structure that is subjected to high temperature. The FGM was first developed by Japanese scientists in the late 1980s, and has gained increasing popularity in recent years.

The finite element method (FEM) has become a mainstream numerical method in engineering due to its maturity and many powerful commercial software packages available. However, the existence of elements in the FEM sometimes embarrasses users. To avoid the use of elements, many sundry mesh-free methods have been developed recently. One common feature of the mesh-free methods is that displacement approximation is implemented based on a set of scattered nodes instead of elements in the influence domain, thus the mesh-free methods can avoid the disadvantages that arise in the FEM from the use of elements. Among the mesh-free methods, the element free Galerkin (EFG) method based on the moving least squares (MLS) approximation technique has the following advantages: (i) the shape function and its derivatives are smooth enough through the choice of appropriate weight function; (ii) the accuracy of the EFG results can reach high enough as desired for engineering application; (iii) the EFG results are very stable.

For the plates that are subjected to non-uniform loads, the distribution of stresses in the plate domain changes largely. The solution of pre-buckling stresses of such plates is not easy to obtain for buckling problem. Seldom related literatures can be found. Liew and Chen [1] employed the mesh-free method based on the radial basis function and two-step method to obtain buckling loads for rectangular and circular plates subjected to concentrated and partial uniform loads.

This paper develops the EFG method for buckling analysis of the piezoelectric FGM plates that are subjected to non-uniformly distributed loads, electricity and heat. A two-step solution procedure is also taken to implement the numerical computation. A variational form of static system equation is established based on a two-dimensional (2D) elastic plane stress problem in the first step, and the variational equation is discretized through displacement approximation. The pre-buckling stresses of the plane elasticity problem are obtained in this step when the plate is subjected to non-uniformly distributed loads. A variational form of the system equation is established based on the Mindlin plate theory in the second step. The variational form is discretized into an eigenvalue equation. The static buckling loads and buckling temperatures of such plates are obtained in this step. Buckling of the piezoelectric FGM plates, with or without electricity and temperature changes, and the pre-buckling stresses resulted from non-uniform loads, is studied.

2 Moving least squares (MLS) technique

The moving least squares (MLS) technique is used for displacement approximation. The elements and their connectivity are avoided. Only an array
of scattered nodes in the domain under consideration is required for such an approximation. Thus the approximation technique is flexible. The number and location of nodes can be handled manually as user’s desire. Here we briefly introduce the generation of displacement approximation based on the MLS technique.

The MLS approximation $u^h(x)$ of a displacement function $u(x)$ at a point $x$ in the problem domain is defined by a polynomial function as

$$u^h(x) = \sum_{j=1}^{m} p_j(x) a_j(x) = p'(x) a(x)$$ (1)

where $p(x)$ is a complete basis whose $m$ components are monomials. $a(x)$ is the vector of coefficients. For buckling problem of plate structures, the dominant displacements are deflections caused by bending of plate structures. Hence we adopt a relatively high-order quadratic polynomial basis.

The unknown coefficients $a(x)$ are obtained by minimizing a weighted residual function:

$$J = \sum_{i=1}^{\pi} w(x - x_i)[p'(x) a(x) - u_i]^2$$ (2)

where $\pi$ is the number of all nodes in the domain under consideration for the point $x$, $w(x - x_i)$ is a weight function of compact support, $u_i$ is a nodal parameter. The quartic spline weight function and its first and second derivatives are continuous over the entire support domain and vanish at its support limit. For our application, we adopt the well-documented quartic spline weight function.

Substituting the obtained $a$ into eqn (1) yields

$$u^h(x) = \sum_{i=1}^{\pi} \phi_i(x) u_i$$ (3)

where $\phi_i$ is the shape function. The shape function does not satisfy the Kronecker delta criterion. Therefore, the MLS technique is an approximation rather than interpolation of a displacement. This MLS property makes the imposition of essential boundary conditions more complicated than the FEM.

3 Governing equations

3.1 Thermo-electro-mechanical FGM plate

A plate, shown in fig. 1, is considered to be made of functionally graded materials (FGMs) through the thickness. The FGM mechanical property is assumed to be linear elastic. The FGM plate varies smoothly and continuously
through the thickness from pure metal of the top surface to pure ceramic of the bottom surface. The volume fractions of the un-symmetrically distributed materials are assumed to be

\[ V_c(z) = \left[\frac{(h + 2z)}{(2h)}\right]^n, \quad V_m(z) = 1 - \left[\frac{(h + 2z)}{(2h)}\right]^n \]  

Equation (4)

where \( V_c(z) \) and \( V_m(z) \) are the ceramic volume fraction and the metal volume fraction respectively, and \( n \) is the non-negative volume fraction exponent. The effective Young’s modulus \( E \), Poisson’s ratio \( \nu \) and thermal expansion coefficient \( \alpha \) can be expressed as

\[ E = E_m + (E_c - E_m)\left[\frac{(h + 2z)}{(2h)}\right]^n, \quad \nu = \nu_m + (\nu_c - \nu_m)\left[\frac{(h + 2z)}{(2h)}\right]^n, \]

\[ \alpha = \alpha_m + (\alpha_c - \alpha_m)\left[\frac{(h + 2z)}{(2h)}\right]^n \]  

Equation (5)

3.2 Pre-buckling stress solution

For the plate that is subjected to non-uniformly distributed in-plane edge loads, the pre-buckling stresses can be calculated by taking the plate as a 2D elastic plane stress problem. The essential boundary conditions don’t need to be considered for the calculation of the pre-buckling stresses because the plate is balanced by the external in-plane edge loads. The variational formulation of the system is expressed as [2].

Figure 1: A FGM plate.
\[ \int_{\Omega} \delta \varepsilon' \sigma' dV - \int_{\Gamma} \delta \mathbf{u}' \mathbf{T} d\Gamma = 0 \quad (6) \]

where \( \varepsilon \) is the vector of strains, \( \sigma \) is the vector of stresses, \( \mathbf{u} \) is the vector of displacements, and \( \mathbf{T} \) is the vector of in-plane edge loads per unit length.

According to eqn (3), the vector of displacements \( \mathbf{u} \) is approximated. Substituting the approximated displacement vector \( \mathbf{u} \) into the variational formulation of eqn (6) gives the discrete system equation as

\[ \mathbf{KU} = \mathbf{f} \quad (7) \]

where \( \mathbf{U} \) is the vector of displacements of all nodes \( \Omega \) in the entire problem domain, \( \mathbf{K} \) is the stiffness matrix that is assembled using the nodal stiffness matrix and \( \mathbf{f} \) is the force vector that is assembled using the nodal force vector.

The displacements of the plate based on the 2D plane stress problem can be obtained by solving eqn (7). Then, the stresses at any point can be obtained.

### 3.3 Buckling solution of the thermo-electro-mechanical FGM plate

For buckling analysis of a plate, the variational formulation with penalty coefficients of the plate can be written as

\[ \int_{\Omega} \delta \varepsilon' \sigma' dV + \int_{\Omega} \delta \varepsilon_n' \tau_n dV + \delta \int_{\Gamma} (\mathbf{u} - \mathbf{u})^T \beta (\mathbf{u} - \mathbf{u}) d\Gamma = 0 \quad (8) \]

where \( \varepsilon' \) is the vector of linear strains, \( \sigma' \) is the vector of linear stresses, \( \varepsilon_n \) is the vector of nonlinear strains, \( \tau_n \) is the vector of stresses corresponding to \( \varepsilon_n \), \( \beta \) is the diagonal matrix with diagonal elements of big number defined by user, \( \mathbf{u} \) is the boundary displacements, and \( \tilde{u} \) is the prescribed boundary displacements.

Based on the Mindlin plate assumption, the displacements of the plate can be written as

\[ u(x, y, z) = u_0(x, y) + z \varphi_x(x, y), \quad v(x, y, z) = v_0(x, y) + z \varphi_y(x, y) \]

\[ w(x, y, z) = w_0(x, y) \quad (9) \]

where \( u_0 \), \( v_0 \), and \( w_0 \) are the displacements and transverse deflection of a point on the plate mid-plane, and \( \varphi_x \) and \( \varphi_y \) denote the rotations about the \( y \)-axis and \( x \)-axis, respectively.

Suppose that both the FGM and the piezoelectric material are linear elastic throughout the deformation. The relationship of the linear stresses \( \sigma' \) and the
linear strains $\varepsilon^l$ for the FGM plate with piezoelectric material, taking into account the piezoelectric and thermal effects, is given by

$$\sigma^l = D_l(\varepsilon^l - \Delta T\bar{u}) - \bar{c}\bar{E}$$

(10)

where $D_l$ is the elastic rigidity matrix of the plate, $\Delta T$ is the temperature change, $\bar{u}$ is the matrix of thermal expansion coefficients, $\bar{c}$ is the matrix of the piezoelectric stiffness, and $\bar{E}$ is the vector of electric fields. We can assume that

$$\begin{bmatrix} E_x & E_y & E_z \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & V_a/h_a \end{bmatrix}^T$$

(11)

where $V_a$ is the applied voltage through the thickness of the actuators.

The high order terms that are associated with the in-plane displacements in the nonlinear strains $\varepsilon_n$ are neglected in this paper, and $\varepsilon_n$ can be simplified as

$$\varepsilon_n = \begin{bmatrix} \frac{1}{2}(w_x)^2 & \frac{1}{2}(w_y)^2 & w_xw_y & 0 & 0 \end{bmatrix}$$

(12)

The stresses $\tau_n$ can be written as

$$\tau_n = \bar{\sigma}^0 = \lambda_p\bar{\sigma}_p^0 - \lambda_T\Delta T D_l \bar{u} - \bar{c}\bar{E}$$

(13)

The independent variables $u_0$, $v_0$, $w_0$, $\phi_x$, and $\phi_y$, of eqn (9) can be approximated using eqn (3). Substituting the approximated independent variables into the variational formulation of eqn (8) gives the discrete eigenvalue equation for the buckling analysis of the plate as

$$[(K_I + K_E + K_u) + \lambda_p G_p + \lambda_T G_T]Q = R_T + R_E$$

(14)

where $\lambda_p$ and $\lambda_T$ are the eigenvalues corresponding to the applied load and the temperature change respectively, $Q$ is the eigenvectors, and $K_I$, $K_E$, $K_u$, $G_p$, and $G_T$ are the matrices relating to the strain energy, the electric voltage, the penalty constraint, the applied load and the temperature change respectively. Solving eqn (14), we can obtain the buckling loads or the buckling temperatures of the plate.

4 Numerical examples

Rectangular plates, shown in figs. 1 and 2, of the FGM with a mixture of zirconia and aluminum are used for examination of the present mesh-free
method. Material properties of the FGM and piezoelectric materials are listed in table 1. The edges of the plates are $a = b = 10.0$ (m), and the thickness is $h = 0.1$ (m). The plates are subjected to non-uniformly distributed in-plane edge loads, heat and electricity.

![Figure 2: A pair of concentrated loads.](image1)

![Figure 3: Opposite axial loads.](image2)

Two kinds of plates: Isotropic plate (i.e., volume fraction exponent $n = 0$ ) and FGM plate with volume fraction exponent $n = 1$ are considered. The buckling loads and the buckling temperatures of the simply supported and clamped plates are calculated using the present method. The boundary conditions are assumed to be $u = v = w = 0$ for simply supported edges and $u = v = w = \varphi_x = \varphi_y = 0$ for clamped edges. For the calculation of buckling load, the temperature of the plate
keeps unchanged. For the calculation of buckling temperature, no in-plane edge loads are applied. For all examples considered here, the size of quadrilateral influence domain is chosen as 3.9 times the average nodal distance for numerical integration. Six terms’ complete polynomial bases are used. Regularly distributed nodes of $14 \times 14$ in the entire plate domain for displacement approximation and Gauss points of $4 \times 4$ in each background cell for numerical integration are used.

Table 1: Material properties.

<table>
<thead>
<tr>
<th>Material properties</th>
<th>Zirconia</th>
<th>Aluminum</th>
<th>G-1195N</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$ (Gpa)</td>
<td>151.0</td>
<td>70.0</td>
<td>63.0</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>$\alpha \times 10^{-5}$ ($1/0^\circ C$)</td>
<td>1.0</td>
<td>2.3</td>
<td>12.0</td>
</tr>
<tr>
<td>$e_{31}$ ($N/mV$)</td>
<td>-</td>
<td>-</td>
<td>22.8429</td>
</tr>
<tr>
<td>$e_{32}$ ($N/mV$)</td>
<td>-</td>
<td>-</td>
<td>22.8429</td>
</tr>
</tbody>
</table>

4.1 A pair of concentrated loads

A rectangular plate that is subjected to a pair of in-plane concentrated loads, shown in fig 2, is considered. The non-dimensional buckling load parameter is defined as $k_{cp} = P_{cr}b/D_0$, where $P_{cr}$ is the critical loads that are applied to the edges of the plate, and the elastic rigidity is $D_0 = E_c h^3/[12(1-\nu^2)]$. The non-dimensional buckling temperature parameter is defined as $k_{ct} = \Delta T_{ct}\alpha_c \times 10^3$, where $\Delta T_{ct}$ is the critical temperature change, and $\alpha_c$ is the coefficient of thermal expansion.

First, the convergence of the buckling loads is studied. No voltage is applied to the plate. The edges of the plate are simply supported. Several densities of distributed nodes are used. The buckling loads are calculated using the present mesh-free method, and their parameters are listed in table 2. The convergence of the present results is good although it is a little oscillating, and the present converged results for isotropic plate are close to the other available results [3].

Table 2: $k_{cp} = P_{cr}b/D_0$ of the plate (a pair of concentrated loads).

<table>
<thead>
<tr>
<th>Nodes</th>
<th>9x9</th>
<th>10x10</th>
<th>11x11</th>
<th>12x12</th>
<th>13x13</th>
<th>14x14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic (n=0)</td>
<td>25.794</td>
<td>25.726</td>
<td>26.527</td>
<td>25.682</td>
<td>25.572</td>
<td>25.573</td>
</tr>
<tr>
<td>FGM (n=1)</td>
<td>18.654</td>
<td>13.906</td>
<td>19.019</td>
<td>18.630</td>
<td>18.525</td>
<td>18.539</td>
</tr>
</tbody>
</table>

\[\text{Result of Leissa and Ayoub [3] is } \kappa_{cp} = 25.814.\]

Second, the buckling loads and the buckling temperatures of the plate with the change of voltage $V_a$ are calculated. The buckling load parameters and the
buckling temperature parameters are obtained, and their results are listed in tables 3 and 4.

Table 3: \( k_{cp} = P_{cr}b/D_0 \) of the plate (a pair of concentrated loads).

<table>
<thead>
<tr>
<th>( V_a )</th>
<th>Isotropic (n=0)</th>
<th>FGM (n=1)</th>
<th>Isotropic (n=0)</th>
<th>FGM (n=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-500</td>
<td>25.677</td>
<td>18.644</td>
<td>69.997</td>
<td>49.061</td>
</tr>
<tr>
<td>-200</td>
<td>25.615</td>
<td>18.581</td>
<td>69.958</td>
<td>49.019</td>
</tr>
<tr>
<td>0</td>
<td>25.573</td>
<td>18.539</td>
<td>69.932</td>
<td>48.987</td>
</tr>
<tr>
<td>200</td>
<td>25.531</td>
<td>18.498</td>
<td>69.905</td>
<td>48.954</td>
</tr>
<tr>
<td>500</td>
<td>25.469</td>
<td>18.435</td>
<td>69.862</td>
<td>48.900</td>
</tr>
</tbody>
</table>

Table 4: \( k_{ct} = \Delta T_{cr}a_0 \times 10^3 \) of the plate (uniform temperature change).

<table>
<thead>
<tr>
<th>( V_a )</th>
<th>Isotropic (n=0)</th>
<th>FGM (n=1)</th>
<th>Isotropic (n=0)</th>
<th>FGM (n=1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-500</td>
<td>0.1269</td>
<td>0.07998</td>
<td>0.3447</td>
<td>0.2100</td>
</tr>
<tr>
<td>-200</td>
<td>0.1266</td>
<td>0.07897</td>
<td>0.3433</td>
<td>0.2097</td>
</tr>
<tr>
<td>0</td>
<td>0.1264</td>
<td>0.07952</td>
<td>0.3441</td>
<td>0.2096</td>
</tr>
<tr>
<td>200</td>
<td>0.1262</td>
<td>0.07934</td>
<td>0.3439</td>
<td>0.2094</td>
</tr>
<tr>
<td>500</td>
<td>0.1259</td>
<td>0.07906</td>
<td>0.3436</td>
<td>0.2091</td>
</tr>
</tbody>
</table>

Table 5: \( k_{cp} = P_{cr}b/D_0 \) of the simply supported plate (opposite axial loads).

<table>
<thead>
<tr>
<th>( V_a )</th>
<th>( c/a = 0 )</th>
<th>( c/a = 0.25 )</th>
<th>( c/a = 0.5 )</th>
<th>( c/a = 1.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( d/a = 0 )</td>
<td>( d/a = 0.25 )</td>
<td>( d/a = 0 )</td>
<td>( d/a = 0.25 )</td>
</tr>
<tr>
<td>( n=0 )</td>
<td>( n=1 )</td>
<td>( n=0 )</td>
<td>( n=0 )</td>
<td>( n=0 )</td>
</tr>
</tbody>
</table>
All of their parameters for the isotropic and FGM plates with simply supported (SSSS) and clamped (CCCC) edges decrease slightly in a linear form as the voltages increase. Both the buckling load parameters and the buckling temperature parameters for the isotropic plate is bigger than those for the FGM plate, and their parameters for the clamped plate is bigger than those for the simply supported plate.

4.2 Three opposite axial loads

A square plate that is subjected to three opposite axial loads, shown in fig. 3, is considered. The axial loads vary through the change of \( c/a \) and \( d/a \). The plate is subjected to several kinds of voltages. The buckling loads of the plate with simply supported edges are calculated, and the buckling load parameters are listed in table 5. Again, the buckling load parameters decrease linearly with a small rate as the voltages increase. As an example, the buckling load parameters of the simply supported plate with the variation of axial load on the top edge and the fixed voltage \( V_a = -200 \text{ v} \) are calculated using the present mesh-free method, and the results are drawn in fig. 4. It can be found that the buckling load parameters increase as the axial load on top edge changes from concentrated load to uniform load.

![Figure 4](image-url)

Figure 4: \( k_{cp} = P_c r b / D_0 \) of the simply supported plate subjected to opposite axial loads.
5 Conclusions

The present mesh-free method approximates displacements based on a set of scattered nodes instead of elements in the influence domain. Thus it avoids the disadvantages that arise in the FEM from the use of elements. As required for different problems and desired for computational accuracy, the nodal distribution, the density of nodes, the order of polynomial basis and the order of weight function can be easily chosen by users without complicated numerical procedures. Numerical examples show that the present mesh-free method has good accuracy, and is a valuable numerical method to the buckling analysis of the piezoelectric FGM plates

References

