Nonlinear vibrations of axisymmetric circular plates with variable thickness

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Abstract

The BEM is developed for nonlinear vibration analysis of axisymmetric circular plates with variable thickness undergoing large deflections. General boundary conditions are considered which may be also nonlinear. The problem is formulated in terms of displacements. The solution is based on the concept of the analog equation, according to which the two coupled nonlinear differential equations with variable coefficients, pertaining to the inplane radial and transverse deformation, are converted to two uncoupled linear ones of a substitute beam with unit axial and unit bending stiffness, respectively, under fictitious quasi-static load distributions. Numerical examples are presented, which illustrate the method and demonstrate its efficiency and accuracy.

Keywords: circular plate, nonlinear, vibrations, large deflections, variable thickness, boundary elements, analog equation method.

1 Introduction

Although much progress has been made in the plate bending analysis by the BEM, only few articles have been published on the analysis of plates with variable thickness. The reason is that the differential equations, which govern the response of the plate, have variable coefficients and thus no fundamental solution is available to derive the boundary integral equations. The problem becomes much more complicated when large deflections are considered. In this case the governing differential equations are coupled and nonlinear in addition to having variable coefficients. For the nonlinear static problem a BEM solution based on the AEM (Analog Equation Method) for arbitrary shaped plate with variable thickness has been presented by Nerantzaki and Katsikadelis [1]. For
circular axisymmetric plates with variable thickness the nonlinear problem is simplified since the spatial variable is only one, i.e. the radial distance. Although the solution for the circular plate could be obtained as a special case of the arbitrary shaped plate, a solution specified to these plates is of special interest, because of their wide use in modern light structures. Analytical solutions of the governing coupled nonlinear equations are out of question. There are some approximate solutions, most of them based on the Berger’s approximations, which are developed either for a specific thickness variation law or specific boundary conditions [2]. Therefore, recourse to numerical methods was inevitable. The FEM has been employed to the problem at hand. Reddy and Huang [3,4] presented a finite element model, which includes also shear deformation and material orthotropy, and used it to study the static and dynamic response of axisymmetric annular and circular plates with variable thickness. With regard to the BEM, a solution for the vibration problem of plates with uniform thickness was reported by Katsikadelis et al. [5].

This paper is an extension to the nonlinear dynamic problem of the recent work of the authors [6] for the nonlinear static problem of axisymmetric circular plates by BEM. The presented BEM approach is general as it can treat both free and forced vibrations of complete and annular circular plates having arbitrary thickness variation law and subjected to all types of boundary conditions. The problem is formulated in terms of the two displacements, that is the membrane radial displacement and the transverse one, both depending on the unique independent variable, the radial distance. This formulation leads to two coupled nonlinear hyperbolic differential equations with variable coefficients pertaining to the inplane radial and transverse deformation under general boundary conditions, some of which are nonlinear. These equations are solved using an integral equation solution based on the AEM for ordinary differential equations [7]. According to this method the two coupled nonlinear differential equations are replaced by two uncoupled linear ones for a beam with unit axial stiffness and unit bending stiffness, respectively, under fictitious quasi-static load distributions. The fictitious loads are established using the integral formulation of the solution for one-dimensional problems. Subsequently, the displacements as well as their derivatives, hence the stress resultants are computed at any point from the respective integral representations, which are used as mathematical formulae. Several numerical examples are presented, which illustrate the method and demonstrate its efficiency and accuracy.

2 Governing equations

The differential equations can be obtained from the general equations derived by Nerantzaki and Katsikadelis [1] for plates of arbitrary shape and thickness variation law by adding the inertia forces and writing them in polar coordinates and taking into account that they are independent of the angular coordinate. However, it is more convenient to derive them by considering the equilibrium of the plate element of the circular plate (see Fig. 1). Without restricting the generality, we neglect the membrane inertia forces. Thus we have
In the radial direction:

\[(rN_r)_{rr} - N_\theta = p\]  \hspace{1cm} (1)

In the transverse direction:

\[\frac{1}{r} (rQ_r)_{rr} + \frac{1}{r} (rN_r w_r), r + q = \rho hw_{rr}\]  \hspace{1cm} (2)

where \(N_r, N_\theta\) are the membrane forces and \(Q_r\) the shearing force per unit length of circular section of radius \(r\); \(q = q(r, t)\) and \(p = p(r, t)\) are the lateral and membrane external loads; \(w = w(r, t)\) is the transverse displacement; \(h(r)\) the variable plate thickness and \(\rho\) the mass density.

![Figure 1: Plate element and applied forces.](image)

Considering nonlinear kinematical relations by retaining the square of the slope of the deflection surface in the radial direction, the membrane forces \(N_r, N_\theta\) are given as [8]

\[N_r = C (u_r + \frac{1}{2} w_{rr} + \nu \frac{u}{r})\]  \hspace{1cm} (3)

\[N_\theta = C \left[ \frac{u}{r} + \nu \left( u_r + \frac{1}{2} w_{rr} \right) \right]\]  \hspace{1cm} (4)

while the bending moments are

\[M_r = -D \left( w_{rr} + \nu w_r \right)\]  \hspace{1cm} (5)

\[M_\theta = -D \left( \frac{1}{r} w_r + \nu w_{rr} \right)\]  \hspace{1cm} (6)
where \( u = u(r,t) \) is the displacement in the radial direction, \( C = Eh/(1-\nu^2) \) is the variable membrane stiffness with \( E \) being the modulus of elasticity, and \( \nu \) the Poisson’s ratio.

The expression for the shearing force \( Q_r \) is obtained by taking the moments of the element forces with respect to an axis perpendicular to the radius. This yields

\[
Q_r = -D
\left(\nabla^2 w_r - D_{\tau r} \left( w_{rr} + \nu \frac{w_r}{r} \right) \right)
\]

in which \( D = Eh^3/12(1-\nu^2) \) is the variable bending stiffness of the plate and

\[
\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}
\]

is the axisymmetric Laplace operator in polar coordinates.

Substituting eqns (3), (4) and (7) into eqns (1) and (2) we obtain the following equilibrium equations in terms of displacements

\[
C\left( u_r + \frac{1}{2} w_r^2 + \nu \frac{u}{r} \right) + r \left[ C \left( u_r + \frac{1}{2} w_r^2 + \nu \frac{u}{r} \right) \right]_r - C \left[ \frac{u}{r} + \nu \left( u_r + \frac{1}{2} w_r^2 \right) \right] = p
\]

\[
D\nabla^4 w + 2D_{\tau r} \left( w_{rr} + \frac{w_r}{r} \right)_r + \left( D_{\tau r} + \frac{D_r}{r} \right) \left( w_{rr} + \frac{w_r}{r} \right) - (1-\nu) \left[ D_{\tau rr} \frac{w_r}{r} + \frac{D_r}{r} w_{rr} \right] - C \left[ u_r + \frac{1}{2} w_r^2 + \nu \frac{u}{r} \right] w_{rr}
\]

\[
+ \left[ \frac{u}{r} + \nu \left( u_r + \frac{1}{2} w_r^2 \right) \right] \frac{w_r}{r} + \rho h w_{tt} = q
\]

in which

\[
\nabla^4 = \frac{d^4}{dr^4} + 2 \frac{d^3}{r \, dr^3} - \frac{1}{r^2} \frac{d^2}{dr^2} + \frac{1}{r^3} \frac{d}{dr}
\]

is the axisymmetric biharmonic operator in polar coordinates. Note that some differentiations in eqn (9) have not been performed for the sake of conciseness of the expressions.

The boundary conditions for an annular plate with an inner radius \( a \) and an outer radius \( b \) can be written as

\[
\alpha_1 w(a,t) + \alpha_2 V_r(a,t) = \alpha_3, \quad \bar{\alpha}_1 w(b,t) + \bar{\alpha}_2 V_r(b,t) = \bar{\alpha}_3 (12a,b)
\]

\[
\beta_1 w_r(a,t) + \beta_2 M_r(a,t) = \beta_3, \quad \bar{\beta}_1 w_r(b,t) + \bar{\beta}_2 M_r(b,t) = \bar{\beta}_3 (13a,b)
\]
\[ \gamma_1 u(a, t) + \gamma_2 N_r(a, t) = \gamma_3, \quad \gamma_1 u(b, t) + \gamma_2 N_r(b, t) = \gamma_3 \] (14a,b)

where \( V_r \) is the total shearing force including the influence of membrane force is given as

\[ V_r = Q_r + N_r w_r = -\frac{1}{\rho} \int_0^r q \rho d \rho \] (15)

and \( \alpha_k, \beta_k, \gamma_k, \bar{\alpha}_k, \bar{\beta}_k, \bar{\gamma}_k \) \((k = 1, 2, 3)\) are given constants. The above edge conditions are general and they can include elastic support as well as imposed boundary actions and/or displacements. Note that all conventional edge conditions can result from them by specifying the constants appropriately, e.g. for a clamped immovable edge at \( r = a \) it is \( \alpha_1 = 1, \alpha_2 = \alpha_3 = 0, \beta_1 = 1, \beta_2 = \beta_3 = 0, \gamma_1 = 1, \gamma_2 = \gamma_3 = 0 \), while for a hinged edge elastically restrained in the plane of the plate by a spring with stiffness coefficient \( k \) and subjected to an edge moment \( \bar{M} \) it is \( \alpha_1 = 1, \alpha_2 = \alpha_3 = 0, \beta_1 = 0, \beta_2 = 1, \beta_3 = \bar{M}, \gamma_1 = k, \gamma_2 = 1, \gamma_3 = 0 \).

In addition to the boundary conditions the deflection \( w(r, t) \) must satisfy the initial conditions

\[ w(r, 0) = g_1(r) \] (16)
\[ w(r, 0) = g_2(r) \] (17)

where \( g_1 \) and \( g_2 \) are given functions.

The boundary conditions for the complete circular plate result by setting \( a = 0 \) in eqns (12a), (13a) and (14a)

\[ Q_r(0, t) = 0 \] (18)
\[ w_r(0, t) = 0 \] (19)
\[ u(0, t) = 0 \] (20)

The condition (18) is a consequence of eqn (19) and \( N_r(0, t) \) finite. This condition should be expressed in terms of the displacements. For this purpose we write eqn (7) as

\[ -\frac{Q_r}{D} = w_{rrr} + \frac{rw_{rr} - w_r}{r^2} + \frac{3h_r}{h} (w_{rr} + \nu w_{rr}) \] (21)

For \( r = 0 \) the second and third term in the right hand side eqn (21) take the indefinite form \( 0/0 \). Applying L’Hôpital’s rule, we obtain

\[ \lim_{r \to 0} \frac{rw_{rr} - w_r}{r^2} = \frac{w_{rrr}(0, t)}{2}, \quad \lim_{r \to 0} \frac{w_r}{r} = w_r(0, t) \]

Using these limits in eqn (21) we obtain
Therefore, the boundary conditions for the complete circular plate are

\[
\frac{3}{2} w_{rrr} (0) + 3 \frac{h_r (0)}{h(0)} (1 + \nu) w_{rr} (0) = 0, \quad \alpha_1 w (b, t) + \alpha_2 V_r (b, t) = \alpha_3
\]  

(22)

(23a,b)

\[
w_r (0, t) = 0, \quad \beta_1 w_r (b, t) + \beta_2 M (b, t) = \beta_3
\]

(24a,b)

\[
u (0, t) = 0, \quad \gamma_1 u (b, t) + \gamma_2 N_r (b, t) = \gamma_3
\]

(25a,b)

3 The AEM solution

Equations (9) and (10) are coupled nonlinear differential equations. They are solved using the AEM for nonlinear vibration problems as developed by Katsikadelis and Tsiatas [7]. The method for the problem at hand is applied as follows.

Let \( u = u(r, t) \) and \( w = w(r, t) \) be the sought solutions, which are two and four times differentiable, respectively, in \((a, b)\). Noting that eqn (9) is of the second order with respect to \( u \) and eqn (10) of fourth order with respect to \( w \) we obtain by differentiating

\[
u_{rr} = b_1 (r, t), \quad w_{rrr} = b_2 (r, t),
\]  

(26)

(27)

where \( b_1, b_2 \) are fictitious loads depending also on time. Eqns (26), (27) indicate that the solution of eqns (9) and (10) can be established by solving eqns (26) and (27) under the same boundary conditions (12)-(14), provided that the fictitious load distributions \( b_1, b_2 \) are first determined. Note that eqns (26) and (27) are quasi-static, that is the time is considered as a parameter.

The fictitious loads are established by developing a procedure based on the boundary integral equation method for one-dimensional problems. Thus, the integral representations of the solutions of eqns (26) and (27) are written as

\[
\nu(r, t) = c_1 r + c_2 + \int_a^b G_1 (r, \xi) b_1 (\xi, t) d \xi,
\]  

(28)

\[
w(r, t) = c_3 r^3 + c_4 r^2 + c_5 r + c_6 + \int_a^b G_2 (r, \xi) b_2 (\xi, t) d \xi,
\]  

(29)

where \( c_i = c_i (t) \) \((i = 1, 2, \ldots 6)\) are arbitrary time dependent integration functions to be determined from the boundary conditions and \( G_i (r, \xi) \) \((i = 1, 2)\) are the fundamental solutions of eqns (26) and (27) given as

\[
G_1 = \frac{1}{2} | r - \xi |
\]  

(30)
$$G_2 = \frac{1}{12} |r - \xi| (r - \xi)^2.$$  

(31)

The derivatives of $u$ and $w$ are obtained by direct differentiation of eqns (28) and (29). Thus,

$$u_{,rr} (r,t) = c_1 + \int_0^l G_{1,rr} (r, \xi) b_1 (\xi, t) \, d\xi ,$$  

(32)

$$w_{,rr} (r,t) = c_3 r^2 + 2 c_4 r + c_5 + \int_0^l G_{2,rr} (r, \xi) b_2 (\xi, t) \, d\xi ,$$  

(33)

$$(r, t) = 6 c_3 + \int_0^l G_{3,rr} (r, \xi) b_2 (\xi, t) \, d\xi ,$$  

(34)

$$(r, t) = 6 c_3 + \int_0^l G_{2,rr} (r, \xi) b_2 (\xi, t) \, d\xi ,$$  

(35)

$$(r, t) = 6 c_3 + \int_0^l G_{2,rr} (r, \xi) b_2 (\xi, t) \, d\xi ,$$  

(36)

$$(r, t) = b_2 (r, t).$$  

(37)

Substituting the derivatives in eqns (9) and (10) yields the necessary equations, from which the fictitious sources $b_1$ and $b_2$ can be determined. This can be implemented only numerically as follows.

The interval $(0, l)$, $l = b - a$, is divided into $N$ equal elements, having length $l / N$. Thus, the integral eqns (28) and (29) are written as

$$u(r,t) = c_1 r + c_2 + \sum_{j=1}^N \int_j G_1 (r, \xi) b_1 (\xi, t) \, d\xi ,$$  

(38)

$$w(r,t) = c_3 r^3 + c_4 r^2 + c_5 r + c_6 + \sum_{j=1}^N \int_j G_2 (r, \xi) b_2 (\xi, t) \, d\xi ,$$  

(39)

where the symbol $\int_j$ indicates the integral on the $j$-element. Subsequently, the fictitious sources are approximated on each integration interval using constant, linear or quadratic variation. In this investigation, the constant element is employed. That is, the fictitious source is assumed constant on the element and its nodal value is placed at the midpoint of the element as illustrated in Figure 2. The integration of the kernels is performed analytically.

$$\text{Nodal points}$$

Figure 2: Discretization of the interval and distribution of the nodal points.
Using this discretization and applying eqns (32)-(39) to the \( N \) nodal points, we obtain

\[
\begin{align*}
\mathbf{u} &= c_1 \mathbf{r}_1 + c_2 \mathbf{r}_0 + \mathbf{G}_1 \mathbf{b}_1 \\
\mathbf{u}_{,r} &= c_1 \mathbf{r}_0 + \mathbf{G}_1, \mathbf{b}_1 \\
\mathbf{u}_{,rr} &= \mathbf{b}_1 \\
\mathbf{w} &= c_3 \mathbf{r}_3 + c_4 \mathbf{r}_2 + c_5 \mathbf{r}_1 + c_6 \mathbf{r}_0 + \mathbf{G}_2 \mathbf{b}_2 \\
\mathbf{w}_{,r} &= 3c_3 \mathbf{r}_3 + 2c_4 \mathbf{r}_2 + c_5 \mathbf{r}_1 + \mathbf{G}_2, \mathbf{b}_2 \\
\mathbf{w}_{,rr} &= 6c_3 \mathbf{r}_3 + 2c_4 \mathbf{r}_2 + \mathbf{G}_2, \mathbf{b}_2 \\
\mathbf{w}_{,rrr} &= 6c_3 \mathbf{r}_3 + \mathbf{G}_2, \mathbf{b}_2 \\
\mathbf{w}_{,rrrr} &= \mathbf{b}_2
\end{align*}
\]

where \( \mathbf{G}_1, \mathbf{G}_1, \ldots, \mathbf{G}_2 \) are \( N \times N \) known matrices, originating from the integration of the kernels \( \mathcal{G}_1(r, \xi), \mathcal{G}_2(r, \xi) \) and their derivatives on the elements; \( \mathbf{u}, \mathbf{u}_{,r}, \ldots, \mathbf{w}_{,rrrr} \) are \( N \times 1 \) vectors including the values of \( \mathbf{u}, \mathbf{w} \) and their derivatives at the nodal points; \( \mathbf{b}_1, \mathbf{b}_2 \) are also \( N \times 1 \) vectors containing the values of the fictitious loads at the nodal points and \( \mathbf{r}_k = \{r_1^k, r_2^k, \ldots, r_N^k\}^T \) are vectors containing the \( k \)-th power of the abscissas of the nodal points.

Finally, collocating eqns (9) and (10) at the \( N \) nodal points and substituting the derivatives from eqns (40)-(47) yields the following system of eqns

\[
\begin{align*}
\mathbf{K}_1(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}) &= \mathbf{p}, \quad (48) \\
\mathbf{M} \ddot{\mathbf{b}}_2 - \mathbf{K}_2(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}) &= \mathbf{q} \quad (49)
\end{align*}
\]

where \( \mathbf{M} \) is a known generalized mass matrix; \( \mathbf{K}_1(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}) \) nonlinear generalized stiffness vectors and \( \mathbf{c} = \{c_1, c_2, \ldots, c_6\}^T \). Eqns (48) and (49) constitute a system of \( 2N \) eqns with \( 2N + 6 \) unknowns. The required six additional eqns result from the boundary conditions. Thus, after substituting the relevant derivatives into eqns (12)-(14), we obtain

\[
\mathbf{f}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}) = \mathbf{0} \quad (50)
\]

Eqn (50) can be used to eliminate the vector \( \mathbf{c} \) from eqns (48) and (49). Thus we obtain

\[
\begin{align*}
\mathbf{K}_1^*(\mathbf{b}_1, \mathbf{b}_2) &= \mathbf{p} \quad (51) \\
\mathbf{M} \ddot{\mathbf{b}}_2 - \mathbf{K}_2^*(\mathbf{b}_1, \mathbf{b}_2) &= \mathbf{q} \quad (52)
\end{align*}
\]
which are the semi-discretized equations of motion of the plate. The associated initial conditions result from eqn (43). They are combined with eqns (16) and (17) yielding $b_2(0)$ and $\dot{b}_2(0)$ after eliminating $c_i$.

Any time step integration method for nonlinear equations of motion can be employed to solve eqn (52). In each iteration for $b_2$ within a time step, the current value of $b_2$ is utilized to update the vectors $b_1$ on the basis of eqn (51). This demands the solution of a nonlinear system of algebraic equations. In this paper, the average acceleration time step integration method in conjunction with modified Newton-Raphson method was employed to solve eqn (52) and the results were cross-checked by a time step integration method based on the analog equation method [9]. Once the vectors $b_1, b_2, c$ are computed, the displacements vectors $u, w$ and their derivatives at any instant $t$ are evaluated from eqns (40)-(47).

4 Numerical examples

On the base of the procedure described in previous section a FORTRAN program has been written for nonlinear dynamic analysis of axisymmetric circular plates with arbitrary thickness and load variation laws. In all examples the results have been obtained using $N = 30$ elements.

![Figure 3: Central deflection versus time in a plate with simply supported movable edge and linear thickness variation.](image)

4.1 Example 1: Linear thickness variation. Simply supported movable edge

The free vibrations of a circular plate with linear thickness variation law $h = h_0(1 - \alpha r / b)$ and simply supported movable edge have been studied; $\alpha$ is
the taper parameter. The edge conditions in this case are \( w(b,t) = 0 \), \( M_r(b,t) = 0 \), \( N_r(b) = 0 \), which result from (23b), (24b) and (25b) for \( \alpha_1 = 1 \), \( \alpha_2 = \alpha_3 = 0 \); \( \beta_1 = \beta_3 = 0 \), \( \beta_2 = 1 \), \( \gamma_1 = \gamma_3 = 0 \), \( \gamma_2 = 1 \). The plate is given an initial deflection \( g_1(r) = h_0 w_1(r) \), while \( g_2(r) = 0 \), where \( w_1(r) \) is the normalized first mode profile of the linear vibration analysis of the same plate. The variation of the central deflection versus time for \( \alpha = 0.5 \) is shown in Fig. 3, from which the nonlinear vibration period is found \( T_{nl} = 3.14 \). Taking into account that the corresponding period of the linear vibration is \( T_{lin} = 5.4771 \) [10], we have \( T_{nl} / T_{lin} = 0.5733 \).

4.2 Example 2: Linear thickness variation. Clamped and hinged edge.

The free vibrations of a circular plate with linear thickness variation law \( h = h_0(1 - \alpha r / b) \) and two types of support condition: (i) clamped edge \( (w(b,t) = 0, \ w_r(b,t) = 0, \ u(b) = 0) \) and (ii) hinged edge \( (w(b,t) = 0, \ M_r(b,t) = 0, \ u(b) = 0) \) have been studied. The ratio \( T_{nl} / T_{lin} \) for various values of the taper parameter \( \alpha \) and central deflection to thickness ratio \( c = w_0 / h_0 \) is presented in Table 1 and Table 2 as compared with those available in the literature. To obtain the period of the nonlinear vibration the plate was subjected to an initial deflection \( g_1(r) = c h_0 w_1(r) \), while \( g_2(r) = 0 \), where \( w_1(r) \) is the corresponding normalized first mode profile of the linear vibration analysis of the same plate. The period of the linear vibration was evaluated using the method described in [10].

<table>
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<tr>
<th>( \alpha )</th>
<th>( c )</th>
<th>( T_{lin} )</th>
<th>Clamped edge</th>
<th>Ref.[3]</th>
<th>Hinged edge</th>
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Table 1: Values of \( T_{nl} / T_{lin} \) in a plate with \( h = h_0(1 - \alpha r / b) \).
Table 2: Values of $T_{nl} / T_{lin}$ in a plate with uniform thickness ($\alpha = 0$).

<table>
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<th>$c$</th>
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<th>Simply supported movable edge</th>
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<td>3.938 0.964 0.966</td>
</tr>
<tr>
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<td>0.940 0.941</td>
</tr>
<tr>
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<td>0.861 0.861</td>
<td>0.914 0.912</td>
</tr>
</tbody>
</table>

4.3 Example 3: Exponential thickness variation. Hinged edge

The free vibrations of a circular plate with exponentially varying thickness, $h = h_0 e^{-\beta (r/b)}$, and hinged edge have been studied. The edge conditions in this case are $w(b,t) = 0$, $M_r(b,t) = 0$, $u(b,t) = 0$. The plate is given an initial deflection $g_1(r) = h_0 w_1(r)$, while $g_2(r) = 0$, where $w_1(r)$ is the corresponding normalized first mode profile of the linear vibration analysis of the same plate. The variation of the central deflection versus time for a plate with $\beta = -0.6$ is shown in Fig. 4 as compared with that for a plate with uniform thickness having the same volume and support conditions. The computed periods of the non linear vibration are $T_{nl}^{exp} = 2.863$ and $T_{nl}^{unif} = 2.961$, respectively. The corresponding periods of the linear vibrations are [10] $T_{lin}^{exp} = 5.457$ and $T_{lin}^{unif} = 5.819$, Thus we obtain the ratios $T_{nl}^{exp} / T_{lin}^{exp} = 0.525$ and $T_{nl}^{unif} / T_{lin}^{unif} = 0.5091$.

![Figure 4: Central deflection versus time in plate with hinged support. Exponential and uniform thickness variation.](image_url)
5 Conclusions

A new solution technique is presented for nonlinear vibration analysis of axisymmetric circular plates with variable thickness. The method is based on the concept of the analog equation which converts the governing coupled nonlinear equations with variable coefficients to two uncoupled linear quasi-static ones pertaining to the axial and transverse deformation of a beam with unit axial and flexural stiffness. The substitute problems are solved using the one dimensional BEM. All types of boundary conditions treated for both annular and complete circular plates. The accuracy of the results is very good.

References


[10] Nerantzaki, M.S., Free vibrations of circular plates with axisymmetric thickness variation (to be published).