Boundary element formulations for stochastic flow in semi-confined aquifers with random boundary conditions

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Abstract

A boundary element technique is developed to solve steady state flow in a shallow aquifer with random boundary conditions. The aquifer is bounded by a leaky aquifer above, and an impervious layer below. The flow problem is formulated in terms of a stochastic boundary value problem involving a Helmholtz equation. The moment of unknown random hydraulic head is obtained by application of an expectation operation to the boundary element formulations. The influence of the spatial structure of the boundary conditions on the hydraulic head statistics is discussed. The significance of the presence of the leaky layer on the hydraulic head statistics is also investigated.

Keywords: semi-confined shallow aquifer, stochastic analysis, random boundary conditions.

1 Introduction

The traditional approach to modelling of groundwater systems has been deterministic. If a variable, for instance, has been measured at a few locations, its distribution in space was determined by some kind of smooth interpolation, and then the flow problem was solved by using appropriate differential equations. This approach is open to criticism for the following reasons. The variables of interest do not vary in a regular manner in space and smooth interpolations are not able to reproduce their fluctuation, and in practice, measurements are generally scarce, i.e. they are available only at a few points, and values at other points are subject to uncertainty.
There is a growing recognition among investigators that stochastic approaches provide a rational framework that accounts for complex randomness often encountered in natural groundwater systems. The last two decades have seen rapid developments in theoretical research treating groundwater systems in a probabilistic framework which links the kinematics of the groundwater system to field measurable quantities. The application of the boundary element method is very common in solving deterministic groundwater problems Lafe, et al. [5]; Ligget and Liu [7]; Lafe and Cheng [6]; Cheng and Morohunfola [3]. The boundary element method has also been used in solving stochastic groundwater flow problems [Cheng and Lafe [2]; Satish and Zhu [8]; Cheng, et al. [1]; Satish and Zhu [9]. Cheng and Lafe [2] proposed an iterative boundary element technique through the distribution of sources and doublets at random intensities along the boundary. Cheng et al. [1] have extended this approach to include the temporal effect. Serrano and Unny [10] have obtained boundary element solutions for the two-dimensional groundwater flow equation with stochastic free surface boundary conditions by using a formulation of the Ito’s lemma in Hilbert space.

In this paper, a boundary element technique will be developed to deal with steady state flows through a shallow semi-confined aquifer with random boundary conditions. Satish and Zhu [8,9] have developed a perturbation-based boundary element technique for groundwater flows in a semi-confined aquifer to investigate effects of random leakage from a leaky layer, as well as random boundary conditions on groundwater systems. But in their work, the random parameters and boundary conditions have been assumed to be homogeneous and no heterogeneities of random variables have been considered. Heterogeneities generally imply significant and irregular variability of flow and transport properties with space. As a consequence of the variable processes involved in the genesis of permeable earth materials, such complex heterogeneities will always exist. It has been demonstrated that the heterogeneities with a given formation can play an important role in groundwater systems. This paper continues our effort to investigate coupled effects of a leaky layer and randomness of input conditions on groundwater systems by including heterogeneities of the boundary conditions. The formation property parameters (leakage factor and hydraulic conductivity) are still assumed to be constant.

2  Groundwater flow in a semi-confined aquifer

In this study, the flow problems in a shallow semi-confined aquifer are considered. The confining beds of a shallow confined aquifer are never truly impermeable. Here the term ‘confined’ indicates that the leakage through the confining beds is negligibly small. If the leakage cannot be neglected, the aquifer is referred to as a semi-confined aquifer. The term ‘shallow’ means that the aquifer is sufficiently shallow that the resistance to flow in the vertical direction may be neglected (Strack [11]). The simplest case of shallow semi-confined flow occurs if the semi-confined aquifer is horizontal and the head is constant above the upper semi-permeable layer, and the lower boundary of the
aquifer is impermeable, as shown in Fig. 1. The hydraulic conductivity and the thickness of the semi-permeable layer are \( K^* \) and \( H^* \), respectively. The hydraulic conductivity and the thickness of the main aquifer are denoted by \( K \) and \( H \), respectively. \( \phi^* \) represents the constant head in the aquifer above the leaky layer.

![Figure 1: Schematic view of a semi-confined aquifer.](image)

If the hydraulic conductivity of the leaky layer \( K^* \) is assumed to be much less than that of the main aquifer \( K \), then the vertical component of flow in the leaky layer is much larger than the horizontal one. We may view a semi-confined aquifer as a confined one with an infiltration through the upper confining bed, as shown in Fig. 1, and define the discharge vector as follows:

\[
P_i = HQ_i = -KH \frac{\partial \phi}{\partial x_i} \quad (i = 1, 2)
\]  

(1)

where \( P_i \) \( (i = 1, 2) \) are the components of the discharge vector and \( Q_i \) \( (i = 1, 2) \) are the components of the specific discharge, respectively.

Then the continuity equation is

\[
\frac{\partial P_i}{\partial x_i} = K^* \left( \phi^* - \phi \right) / H^*
\]  

(2)

Substituting (1) into (2), one obtains

\[
\frac{\partial}{\partial x_i} \left( K \frac{\partial \phi}{\partial x_i} \right) = -K \lambda^2 (\phi^* - \phi)
\]  

(3)

Defining relative head \( \phi_r = \phi - \phi^* \) and dropping the subscript for simplicity, one obtains

\[
\frac{\partial}{\partial x_i} \left( K \frac{\partial \phi}{\partial x_i} \right) = K \lambda^2 \phi
\]  

(4)
where $\lambda^* = \sqrt{K^*/KH^*}$ is called the leakage factor. It should be noted that $1/\lambda^*$ has the dimensions of length.

In this paper, the main aquifer considered is assumed to be homogeneous, that is, $K$ is a constant. After normalizing the head by a characteristic head $\phi_0$, the coordinates by a characteristic length $L$, and $\lambda^*$ by $L^{-1}$, one simplifies the governing equation for groundwater flow in a shallow semi-confined aquifer, eqn (4), as follows:

$$\nabla^2 \phi - \lambda^2 \phi = 0$$

(5)

As stated previously, $K^*$ is assumed to be much smaller than $K$ and therefore $\lambda$ is a small constant.

Boundary conditions may consist of the following two types:

**Dirichlet type**

$$\phi = \psi(x) \quad x \in \Gamma_\phi$$

(6)

**Neumann type**

$$q = \frac{\partial \phi}{\partial n} = p(x) \quad x \in \Gamma_q$$

(7)

where $n$ is the outward unit normal to the Neumann boundary $\Gamma_q$, $\Gamma_\phi + \Gamma_q = \Gamma = \partial V$ is the whole boundary of the domain of interest $V$ and $\psi(x)$ and $p(x)$ are random functions. It is assumed that the expectations

$$\overline{\psi}(x) = E[\psi(x)] \quad \text{for } x \in \Gamma_\phi$$

(8)

$$\overline{p}(x) = E[p(x)] \quad \text{for } x \in \Gamma_q$$

(9)

and covariance functions

$$c_{\psi\psi}(x, y) = E\left[(\psi(x) - \overline{\psi}(x))(\psi(y) - \overline{\psi}(y))\right] \quad \text{for } x, y \in \Gamma_\phi$$

(10)

$$c_{pp}(x, y) = E\left[(p(x) - \overline{p}(x))(p(y) - \overline{p}(y))\right] \quad \text{for } x, y \in \Gamma_q$$

(11)

and their cross-covariance functions

$$c_{\psi p}(x, y) = E\left[(\psi(x) - \overline{\psi}(x))(p(y) - \overline{p}(y))\right] \quad \text{for } x \in \Gamma_\phi \text{ and } y \in \Gamma_q$$

(12)

$$c_{p\psi}(x, y) = E\left[(p(x) - \overline{p}(x))(\psi(y) - \overline{\psi}(y))\right] \quad \text{for } x \in \Gamma_q \text{ and } y \in \Gamma_\phi$$

(13)

for the random functions $\psi(x)$ and $p(x)$ are known. Here, $E$ denotes an expectation operator. The variances of $\psi(x)$ and $p(x)$ can be defined as follows

$$\sigma^2_{\psi}(x) = E\left[\left((\psi(x) - \overline{\psi}(x))^2\right]\right] \quad \text{for } x \in \Gamma_\phi$$

(14)
\[ \sigma_p^2(x) = E\left[\left((p(x) - \bar{p}(x))^2\right)\right] \quad \text{for } x \in \Gamma_q \] (15)

The stochastic flow problems given previously may be formulated mathematically to determine moments of unknown head and its gradients on the boundary and inside the domain in terms of the given moments (expectation, covariance and cross-covariance functions) on the boundary.

3 Stochastic boundary integral equations

The weighted residual statement for (5) can be written as

\[ \int_{\Gamma} \left[ \nabla^2 \phi - \lambda^2 \phi \right] \omega dV = \int_{\Gamma} (q - p) \omega d\Gamma - \int_{\Gamma} (\phi - \eta) \frac{\partial \omega}{\partial n} d\Gamma \] (16)

where \( \omega \) is a weight function.

If one chooses \( \omega \) as \( \Phi^* \) which satisfies the following equation,

\[ \nabla^2 \Phi^* - \lambda^2 \Phi^* - \delta(x) = 0 \] (17)

then one can obtain

\[ \gamma \phi(x) = \int_{\Gamma} \left[ \phi \frac{\partial \Phi^*}{\partial n} - q \Phi^* \right] d\Gamma \] (18)

where \( \delta(x) \) is a Dirac delta function acting at point \( x \in V \), \( \gamma \) is a coefficient (\( \gamma = 1 \) inside the domain, \( \gamma = 1/2 \) on smooth boundary) and \( \Phi^* \) in (17) is referred to as the fundamental solution. The fundamental solution to these equations has been given (Greenberg [4]) as

\[ \Phi^* = -\frac{1}{2\pi} K_0(\lambda r) \] (19)

where \( r \) is the distance between the base point and a field point and \( K_0 \) is the modified Bessel function of the second kind of order zero.

When \( x \) is inside the domain, it can be shown that

\[ \phi(x) = \frac{1}{2\pi} \int_{\Gamma} \left[ \lambda K_1(\lambda r) \frac{\partial r}{\partial n} \phi + K_0(\lambda r) q \right] d\Gamma \] (20)

\[ \frac{\partial \phi(x)}{\partial x_i} = \frac{1}{2\pi} \int_{\Gamma} \left[ -K_0(\lambda r) - \frac{K_1(\lambda r)}{\lambda r} \right] \frac{\lambda^2 \eta \partial r}{r \partial x_i} \right] \phi d\Gamma + \frac{1}{2\pi} \int_{\Gamma} \left[ -\lambda K_1(\lambda r) \frac{\partial r}{\partial x_i} \right] q d\Gamma \] (21)

where \( K_1 \) is the modified Bessel function of the second kind of order one and \( \eta \) is the projection of \( r \) on \( n \).
For the case when \( x \) is on the boundary \( \Gamma \), it can be shown that

\[
-\pi \phi(x) + \int_{\Gamma} \left[ \lambda K_1(\lambda r) \frac{\partial r}{\partial n} \phi \right] d\Gamma = -\int_{\Gamma} K_0(\lambda r) q d\Gamma \tag{22}
\]

Equation (22) relates the boundary values of \( \phi \) and \( q \), while eqns (20) and (21) express the values of \( \phi \) and its gradients at an internal point of interest in terms of the boundary values of \( \phi \) and \( q \).

4 Discretized solutions of the stochastic integral equation and numerical evaluations of moments

4.1 Discretization and solution of the stochastic integral equation

The boundary is discretized into a collection of straight-line elements. The values of \( \phi \) and \( q \) are assumed to be constant on each element. Under this discretization scheme, (22) becomes

\[
\left( -\pi \delta_{ij} + C_{ij} \right) \phi_j = D_{ij} q_j \quad (i=1, \ldots, N) \tag{23}
\]

where

\[
C_{ij} = \begin{cases} 
\int_{\Delta\Gamma_j} \lambda K_1(\lambda r) \frac{\partial r}{\partial n} d\Gamma & \text{if } j \neq i \\
0 & \text{if } j = i 
\end{cases} \tag{24}
\]

\[
D_{ij} = -\int_{\Delta\Gamma_j} K_0(\lambda r) d\Gamma \tag{25}
\]

and \( r \) is the distance from point \( i \) to a point on the boundary element \( \Delta\Gamma_j \), and \( N \) is the total number of boundary elements. The integrations for every boundary element integral have been carried out by using Gaussian quadrature.

The linear algebraic equation system (23) can also be expressed as follows

\[
Au = Bv \tag{26}
\]

where \( u \) and \( v \) are the unknown vectors denoting unknown and known functions on the boundary. Their components are either \( \phi \) or \( q \) depending on whether the boundary element is located on \( \Gamma_\phi \) or \( \Gamma_q \). If \( i \in \Gamma_\phi \), then \( u_i = \phi_i \) and \( v_i = \phi_i \). If \( i \in \Gamma_q \), then \( u_i = q_i \) and \( v_i = q_i \).

The components of coefficient matrices \( A \) and \( B \) are given as follows

If \( j \in \Gamma_\phi \), \( A_{ij} = D_{ij} \) and \( B_{ij} = -\pi \phi_j + C_{ij} \)

If \( j \in \Gamma_q \), \( A_{ij} = \pi \phi_j - C_{ij} \) and \( B_{ij} = -D_{ij} \)

The solution to the system (26) is

\[
u = Hv \tag{27}\]

where \( H = A^{-1}B \).
4.2 Numerical evaluations of moments

After the expression for the boundary solutions has been constructed, the solutions of \( \phi \) and its gradient at any internal point of interest can be expressed in terms of the boundary values of \( \phi \) and \( q \).

From (20), one obtains

\[
\phi(x) = \frac{1}{2\pi} \int_{\Gamma_\phi} \lambda K_1(\lambda r) \frac{\partial r}{\partial n} \phi d\Gamma + \frac{1}{2\pi} \int_{\Gamma_q} K_0(\lambda r) q d\Gamma \\
+ \frac{1}{2\pi} \int_{\Gamma_\phi} \lambda K_1(\lambda r) \frac{\partial r}{\partial n} q d\Gamma + \frac{1}{2\pi} \int_{\Gamma_q} K_0(\lambda r) p d\Gamma
\]

(28)

After discretization, one obtains

\[
\phi_i = \frac{1}{2\pi} \sum_{j=1}^{N_\phi} \left[ \int_{\Delta \Gamma_j} \lambda K_1(\lambda r) \frac{\partial r}{\partial n} d\Gamma \right] \phi_j + \frac{1}{2\pi} \sum_{j=1}^{N_q} \left[ \int_{\Delta \Gamma_j} \lambda K_0(\lambda r) d\Gamma \right] q_j \\
+ \frac{1}{2\pi} \sum_{j=1}^{N_\phi} \left[ \int_{\Delta \Gamma_j} \lambda K_1(\lambda r) \frac{\partial r}{\partial n} d\Gamma \right] \psi_j + \frac{1}{2\pi} \sum_{j=1}^{N_q} \left[ \int_{\Delta \Gamma_j} K_0(\lambda r) d\Gamma \right] p_j
\]

(29)

where \( N_\phi \) is the total number of boundary elements of \( \Gamma_\phi \), \( N_q \) is the total number of boundary elements of \( \Gamma_q \) and \( N_\phi + N_q = N \).

For every boundary element, one needs to calculate the following integrals

\[
S_{ij} = \frac{1}{2\pi} \int_{\Delta \Gamma_j} \lambda K_1(\lambda r) \frac{\partial r}{\partial n} d\Gamma
\]

(30)

\[
T_{ij} = \frac{1}{2\pi} \int_{\Delta \Gamma_j} K_0(\lambda r) d\Gamma
\]

(31)

The internal solutions can then be expressed as

\[
\phi_i = G_{ij} v_j
\]

(32)

where \( G_{ij} \) can be calculated in the following ways

If \( j \in \Gamma_\phi \), \[ G_{ij} = S_{ij} + T_{ik} H_{kj} \]

(33)

If \( j \in \Gamma_q \), \[ G_{ij} = T_{ij} + S_{ik} H_{kj} \]

(34)

The internal solutions for \( \partial \phi / \partial x_1 \) and \( \partial \phi / \partial x_2 \) in terms of input boundary values can be constructed similarly by using formula (21).

After expressing solutions in terms of input boundary values, moments of output can be obtained as follows:

\textit{Expectations:}

Taking expectation on both sides of eqn (27), the boundary expectation can be obtained as follows
the internal head expectation is

\[ \bar{\phi}_i = G_{ij} \bar{v}_j \]  

\( \text{(36)} \)

**Covariances:**

The fluctuations of boundary values and internal head are

\[ u'_i = H_{ij} v'_j \]  

\( \text{(37)} \)

\[ \phi'_i = G_{ij} v'_j \]  

\( \text{(38)} \)

The boundary covariances can be obtained as

\[ c_{uu}(i, j) = E(u_i u_j) = H_{ik} H_{jm} c_{vv}(k, m) \]  

\( \text{(39)} \)

where \( c_{vv}(k, m) \) is the input covariance of known boundary values.

The internal covariances are

\[ c_{\phi \phi}(i, j) = E(\phi_i \phi_j) = G_{ik} G_{jm} c_{vv}(k, m) \]  

\( \text{(40)} \)

The expectations and covariances of head gradients can be constructed similarly.

5 **Numerical example: semi-confined aquifer bounded by streams with random stages**

As an example, we consider the groundwater flow problem in a semi-confined aquifer bounded between two rivers. The boundary conditions are that the heads along the riverbanks are specified by random river stages \( \phi_1 \) and \( \phi_2 \) respectively, whose expectations \( \bar{\phi}_1, \bar{\phi}_2 \), variances \( \sigma_1^2, \sigma_2^2 \), and correlation coefficient \( \rho \) are known and are sketched in Fig. 2. It should be noted that \( \phi_1 \) and \( \phi_2 \) are the river stages relative to the head above the leaky layer. This problem is one dimensional and exact solutions can be found and are given in the Appendix.

This problem has been also solved by the boundary element method presented previously in two dimensions as shown in Fig. 2. Here, \( R \) is arbitrary and its value does not affect the boundary element solution. Along \( x_1 = 0 \), covariance function \( c_{\phi \phi}(x'_2, x'_2) = \sigma_1^2 \), and along \( x_1 = 1 \), \( c_{\phi \phi}(x''_2, x''_2) = \sigma_2^2 \) are prescribed, where \( x'_2 \) and \( x''_2 \) are two arbitrary \( y \) coordinates located on the same segment. If \( x'_2 \) and \( x''_2 \) are located on different segments along \( x_1 = 0 \) and \( x_1 = 1 \), \( c_{\phi \phi}(x'_2, x''_2) = \rho \sigma_1 \sigma_2 \) are prescribed, where \( \rho \) is the correlation coefficient of the two river stages. On the boundary \( x_2 = 0 \) and \( x_2 = R \), the deterministic condition \( q = 0 \) is used. All covariance functions \( c_{\phi q}, c_{q \phi}, \) and \( c_{qq} \) associated with these two sides vanish. The governing equation, along with the boundary condition
input described previously are solved numerically using 48 boundary elements (12 on each side).

Figure 2: Definition sketch of a semi-confined aquifer between two rivers with random stages.

Figure 3: Head expectation for two-river problem.

To investigate the influence of leakage factor and river stage correlation, we assume that the mean stage at $x_1 = 0$, $\bar{\phi}_1$ is equal to 1 and that at $x_1 = 0$, $\bar{\phi}_2$ is equal to 0.5 and also that the standard deviation of river stage at $x_1 = 0$ is twice that at $x_1 = 1$, that is, $\sigma_1 = 2\sigma_2$. In Fig. 3, the head expectation $\overline{\phi}(x_1)$ is plotted versus $x$ for three different values of $\lambda$, 0.1, 0.5, and 1. Results show that the influence of leakage on the head gradient is greater. For an aquifer confined between two rivers, i.e. $\lambda = 0$, the head expectation is nearly a straight line. The
The head variance \( \sigma_{\phi}^2(x_1) = c_{\phi\phi}(x_1, x_1) \), normalized by \( \sigma_1^2 \), is plotted in Fig. 4, against \( x_1 \) for different correlations of river stage when \( \lambda = 1 \). From these results, it can be observed that the head variance increases with the increase in river stage correlation. For a confined aquifer, Cheng and Lafe [1991] reported a similar increase in head variance with increase in river stage correlation.

The numerical solutions of correlations of the head between the boundary value and a point \( x_1 \) in the aquifer are also calculated and compared with the exact solution. Figure 5 show the normalized head correlation \( c_{\phi\phi}(0, x_1)/\sigma_1^2 \) as function of \( x_1 \) for various river stage correlation values. It is observed that the head correlation decreases as the river stages become less correlated.

For all the results presented, close agreement between the numerical results and the exact solution may be observed.

Figure 4: Head variance for two-river problem when \( \lambda = 1.0 \).

Figure 5: Head correlation for two-river problem when \( \lambda = 1.0 \).
6 Conclusions

A boundary element method has been presented in this paper for the solution of stochastic groundwater flow problems in a semi-confined aquifer subject to random boundary conditions. Aquifers are semi-confined if leakage from above the semi-permeable layer is not negligible and there is little or no flow in the aquifer above the semi-permeable layer. A boundary integral equation is derived to relate unknown boundary quantities and known boundary conditions. From this equation, the expectations and covariance functions of boundary solutions have been constructed in terms of input boundary value statistics by taking expectation of appropriate combinations of the resulting algebraic linear system. The internal solutions have been obtained by boundary integrals. By application of expectation operation, moments of unknown random functions have been obtained. Numerical examples for one-and two-dimensional flow problems are examined and presented. For one-dimensional problems, the numerical solutions have been compared favourably with the exact solutions both for expectations and covariance functions of unknown random head and its gradients. It has been shown that the leakage factor reduces head variances and stronger head correlation on the boundary results in larger head variances.

The derivation of governing equation (5) assumes a small, deterministic and homogeneous leakage factor along with deterministic and homogeneous hydraulic conductivity of the main aquifer. The cases involving random leakage factors and hydraulic conductivities deserve future studies.

References


