The Analytic Element Method and supporting GIS geodatabase model

D. R. Steward[1], P. Le Grand[2] & E. A. Bernard[3]
[1] *Kansas State University, Department of Civil Engineering, U.S.A.*
[2] *Ecole des Mines, Saint-Etienne, France*
[3] *Kansas State University, Department of Landscape Architecture/Regional and Community Planning, U.S.A.*

Abstract

This manuscript presents an overview of the Analytic Element Method, and illustrates how this mathematical technique is ideally suited to utilization within a GIS geodatabase model. The Analytic Element Method contains a set of analytic elements that exactly satisfy the governing partial differential equation and represent flow associated with a point, line or polygon. Elements are superimposed to simulate local and regional flow within an infinite domain. A GIS geodatabase model is presented here, which organizes spatial data in a vector format that relates directly to analytic elements. Scripts have been developed to automate the creation of groundwater models from the GIS geodatabase using the computer model MLAEM (Multi-Layer Analytic Element Model). An example is presented to illustrate the efficacy of this approach.
Keywords: Analytic Element Method, groundwater, ground water, Geographic Information System, GIS, geodatabase.

1 Introduction

The Analytic Element Method was developed by Otto D. L. Strack and his students at the University of Minnesota over the past 30 years. This mathematical representation of groundwater flow has been published in books by Strack [8] and Haitjema [3], and the most recent advances in the methodology was recently summarized in Strack [9]. The AEM began as a means of solving problems with idealized shapes (e.g., flow around cylindrical objects) and has evolved into a robust solution technique capable of modeling local detail within regional models the size of nations (e.g., NAGROM in The Netherlands; www.riza.nl).

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GIS geospatial technology has also seen recent technological advances. Geodatabase models have been developed to model surface water hydrology by Maidment [5], and are being integrated with groundwater components by Strassberg and Maidment [10]. To date, groundwater models are limited to raster data organization using Finite Difference and Finite Element Methods.

This manuscript presents the methodology by which groundwater models based upon the Analytic Element Method can be integrated with GIS geodatabase models using vector datasets, implementing and extending an approach put forth by Steward and Bernard [7]. First, an overview of the Analytic Element Method is presented. This is followed by development of an AEM Geodatabase model and an illustrating application.

2 Analytic Element Method

2.1 Methodology

The Analytic Element Method is a technique for solving partial differential equations over domains of either infinite or finite extent. This method obtains a solution using the following steps:

1. Discretize boundaries of features within the domain into a set of analytic elements with prescribed geometry.
2. Formulate closed-form solutions for each analytic element with two properties: a) they exactly satisfy the partial differential equation, and b) they produce a withdrawal or velocity along the element.
3. Solve for unknown strength coefficients to satisfy boundary conditions.
4. Evaluate the mathematical expressions for all analytic elements to obtain the potential and vector field.

2.2 Groundwater governing equations

Groundwater flow satisfies Darcy's Law, which may be written as follows for two-dimensional Dupuit flow in an aquifer with piecewise constant hydrogeologic properties (following Strack [8]),

$$\vec{Q} = -\vec{\nabla}\Phi \quad , \quad \Phi = \begin{cases} \dfrac{k(\phi - B)^2}{2} & (\phi - B) < H \\ kH(\phi - B) - \dfrac{kH^2}{2} & (\phi - B) >= H \end{cases} \tag{1}$$

where $\vec{Q}$ is the discharge per unit width of aquifer, Φ is the potential for $\vec{Q}$, k is the hydraulic conductivity, B is the base elevation, H is the aquifer thickness, and ϕ is the groundwater head. Darcy's Law together with continuity of flow gives

$$\nabla^2\Phi = -R + \frac{1}{\alpha}\frac{\partial\Phi}{\partial t} \tag{2}$$

where R is the specific discharge of recharge and α is the aquifer diffusivity. For two-dimensional irrotational, divergence-free flow, this equation reduces to the Laplace Equation, which may be represented in complex form as

$$\frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} = 0 \quad , \quad \Omega = \Phi + \mathrm{i}\Psi \tag{3}$$

where Ω is a complex potential with real part Φ and an imaginary part equal to the stream function Ψ.

The Analytic Element Method in groundwater flow formulates a set of elements that exactly satisfies (2) or (3) and generates flow associated with the geometry of a particular hydrogeological feature (Strack [8, 9]; Haitjema [3]). The mathematical equations for all elements are superimposed to provide a comprehensive potential for groundwater flow within an aquifer. This comprehensive potential may be evaluated at any point to provide the flow rate and head using (1).

2.3 AEM equations for steady two-dimensional flow

Analytic elements have the geometry of points, lines and polygons. The simplest point element is obtained from the fundamental solution of the two-dimensional Laplace equation, (3);

$$\Omega = \frac{1}{2\pi} \ln\left(z - z_0\right) \tag{4}$$

where $z = x + \mathrm{i}y$. This point-sink element generates a vector field corresponding to a unit withdrawal of flux at z_0. The derivatives of (4) in the y-direction

$$\Omega = \frac{\mathrm{i}}{2\pi}\frac{1}{z - z_0} \tag{5a}$$

and in the x-direction

$$\Omega = \frac{1}{2\pi}\frac{1}{z - z_0} \tag{5b}$$

generates point-doublet elements with vector fields associated with an infinite velocity at z_0 aligned in the direction of differentiation.

Analytic elements with the geometry of lines may be obtained by integrating point elements along a line. This is done here for a straight line segment that lies along the x-axis from $x = -L$ to $x = L$ (Steward & Jin [6]). Integrating a distribution of point-sinks, (4), along this line

$$\Omega = \int_{-L}^{L} S(\tilde{x}) \frac{1}{2\pi} \ln\left[(x - \tilde{x}) + \mathrm{i}y\right] \mathrm{d}\tilde{x} \tag{6}$$

gives an analytic element that is called a line-sink (Strack, [8]) or a single layer. Integrating a distribution of point-doublets oriented perpendicular to the line, (5a),

$$\Omega = \int_{-L}^{L} S(\tilde{x}) \frac{\mathrm{i}}{2\pi} \frac{1}{(x - \tilde{x}) + \mathrm{i}y} \mathrm{d}\tilde{x} \tag{7a}$$

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gives a line-doublet (Strack, [8]) or a double layer. Integrating point-doublets oriented parallel to the line, (5b),

$$\Omega = \int_{-L}^{L} S(\tilde{x}) \frac{1}{2\pi} \frac{1}{(x-\tilde{x}) + \mathrm{i}y} \mathrm{d}\tilde{x} \tag{7b}$$

gives a line-dipole (Strack, [8]). These line elements withdraw a flux along the line, (6), or generate a velocity perpendicular to, (7a), or parallel to, (7b) the line.

Analytic elements with the geometry of a polygon are obtained using a combination of mathematical equations within the polygon and one or more of the line elements (6), (7a), and (7b) along the boundary of the polygon. For example, an area-sink analytic element corresponding to a polygon over which a constant rate of recharge R_0 occurs could me modeled using (following Strack [8])

$$\Phi = \begin{cases} -R_0 \dfrac{x^2}{2} & z \in \text{polygon} \\ 0 & z \notin \text{polygon} \end{cases} \tag{8}$$

While this equation satisfies the governing equation (2), it generates jumps in both the potential and the velocity along the boundary of the polygon. Line-doublets, (7b), which create a jump in potential across a line, are placed on the boundary to counter the jump in potential in (8), thereby satisfying continuity of head. Line-sinks, (6), which create a jump in the normal component of velocity across a line, are placed on the boundary to satisfy continuity of flow.

Similar area-heterogeneity analytic elements exist to reproduce the flow generated by polygons with distinctly different aquifer properties (hydraulic conductivity, base elevation and thickness) than the surrounding media (e.g., see Strack [8]). In this case, continuity of head and flow may be satisfied using line-doublets placed along the boundary.

There is a much easier methodology to obtain the mathematical expressions for analytic elements with the geometry of lines than direct integration of (6), (7a), and (7b). Following Strack [8, 9], the potential for a line-doublet has the following form,

$$\Omega = \frac{\mathrm{i}}{2\pi} S(z) \left[\ln \frac{z+L}{z-L} + f(z) \right] \tag{9}$$

where f is a polynomial function of z. Assuming the strength varies as a polynomial function of distance along the line element (Janković & Barnes [4])

$$S(z) = \sum_{m=0}^{M} S_m \left(\frac{z}{L} \right)^m \tag{10}$$

gives

$$\Omega = \frac{\mathrm{i}}{2\pi} \sum_{m=0}^{M} S_m \left[\left(\frac{z}{L} \right)^m \ln \frac{z+L}{z-L} + f_m(z) \right] \tag{11}$$

The far-field correction terms $f_m(z)$ associated with strength parameter S_m are chosen such that Ω behaves like $1/z$ or lower orders of z at infinity. This function may be obtained using the series expansion (from Abramowitz and Stegun [1])

$$\ln\frac{z+L}{z-L} = \sum_{j=1}^{\infty}\frac{2}{2j-1}\left(\frac{z}{L}\right)^{-2j+1} \tag{12}$$

Thus, Ω has the correct far-field behavior if

$$f_m(z) = -\sum_{j=1}^{\frac{m+1}{2}}\frac{2}{2j-1}\left(\frac{z}{L}\right)^{m-2j+1} \tag{13}$$

which gives the general expression for an m^{th}-order line-doublet

$$\Omega = \frac{\mathrm{i}}{2\pi}\sum_{m=0}^{M} S_m\left[\left(\frac{z}{L}\right)^m \ln\frac{z+L}{z-L} - \sum_{j=1}^{\frac{m+1}{2}}\frac{2}{2j-1}\left(\frac{z}{L}\right)^{m-2j+1}\right] \tag{14a}$$

At large distances from the element where $(z/L)^m \ln[(z+L)/(z-L)]$ and $f_m(z)$ cancel, the complex potential may be obtained from the higher-order terms in the series expansion

$$\Omega = \frac{\mathrm{i}}{2\pi}\sum_{m=0}^{M} S_m \sum_{j=\frac{m+3}{2}}^{\infty}\frac{2}{2j-1}\left(\frac{z}{L}\right)^{m-2j+1} \tag{14b}$$

In practice, this series may be truncated after a few terms, with the number of terms required decreasing at larger distances $|z| \gg L$ from the element.

The mathematical expression for a line-dipole may, of course, be obtained from (14) by dividing the expression by i. Mathematical expressions for line-sinks may be obtained by integrating the expression for a line-dipole with respect to z,

$$\Omega = -\frac{1}{2\pi}\int S(z)\left[\ln\frac{z+L}{z-L} + f(z)\right]\mathrm{d}z \tag{15}$$

Following Strack [8], line-sinks may also be obtained using combinations of line-dipoles with point-sinks, (4), placed at the ends of the lines.

2.4 Associating hydrogeologic features with analytic elements

A list of hydrogeologic features is presented in table 1. Each hydrogeologic feature has the geometry of a point, line or polygon. Likewise, each feature may be represented mathematically using the analytic elements presented earlier.

A groundwater system is composed of a set of aquifer layer through which water moves, which are separated by a set of aquitard layers which retard the movement of water. Heterogeneities have distinctly different aquifer properties (k, B and H)

Table 1: Components of analytic elements in groundwater.

Hydrogeology	Geometry	Mathematics
Aquifer or aquitard layer	polygon	
Heterogeneity	polygon	area-heterogeneity
	line	line-doublet or line-dipole
Lake	polygon	area-sink
Recharge	polygon	area-sink
Regional interaction	polygon	Laurent series
River	line	line-sink
Well	point	point-sink

from the rest of the layer and may have the geometry of a polygon or, in the limit of very thin features, of a line. Lakes and recharge zones are shaped like polygons; narrow rivers lie along lines; and wells lie at points. Regional interactions outside the local area of interest may be modeled using a Laurent series. Heterogeneities, lakes, recharge, regional interaction, rivers and wells are all associated with the aquifer or aquitard layer in which it resides.

3 AEM Geodatabase Model

The Analytic Element Method naturally associates a mathematical representation of groundwater flow with vector representation of hydrogeologic features using points, lines and polygons (Steward and Bernard [7]). This natural association is exploited here to organize a GIS geodatabase model for the Analytic Element Method in ArcGIS. Python scripts have been developed to automate creation of and visualization of groundwater models using this geodatabase structure. These scripts utilize the groundwater model MLAEM (Multi-Layer Analytic Element Model).

An overview of the AEM geodatabase model is illustrated using the Unified Modeling Language (UML) in figure 1. Following Zeiler [11] and Arctur & Zeiler [2], each box represents a Feature Class (with geometry) or Object Class (without geometry). Relationship classes between feature and objects classes are documented by lines in the UML diagram, which each represent aggregation (———◇), generalization (———▷), association (————), or a link to a comment box (┈┈┈┈). Relationships indicate multiplicities using numbers at the end of each line.

The AEM Geodatabase structure parallels the structure provided within the ArcHydro GIS Geodatabase (Maidment [5]; Strassberg & Maidment [10]), with two major feature datasets related to hydrogeology and modeling. The AEM Geodatabase is, however, different from previous efforts in that this geodatabase

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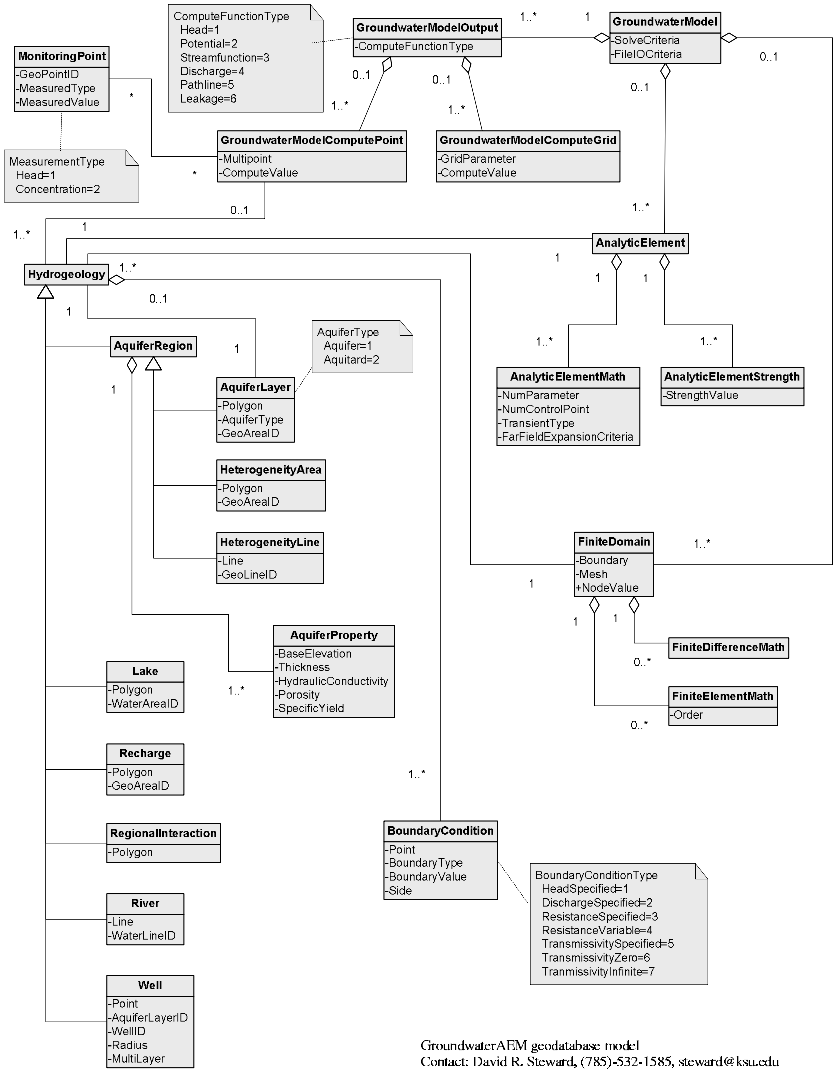

Figure 1: The Analytic Element Geodatabase.

provides a direct link between hydrogeologic vector data and the Analytic Element Method. Each hydrogeology feature is associated with an aquifer layer, a set of boundary conditions, and an analytic element mathematical representation. A groundwater model contains a collection of analytic elements and specifications for model output. While links to groundwater modeling techniques based upon the finite difference and finite element methods are identified, these links have not yet been developed.

4 Application

An example is presented here to illustrate the results obtained using the AEM geodatabase in figure 1 with the groundwater model MLAEM. Model results, in figure 2 are shown near wells for the city of Manhattan, KS (white dots). These wells are located near the confluence of the Big Blue and Kansas Rivers (white lines). Groundwater is illustrated using shaded contours with constant elevation.

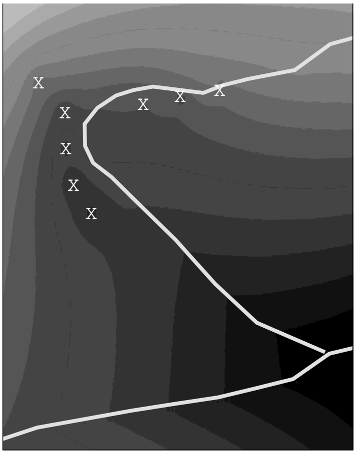

Figure 2: AEM model results.

This model was constructed by first loading the AEM Geodatabase model using GIS data from DASC (www.kgs.ukans.edu), which is the FGDC (Federal Geographic Data Committee) clearinghouse for the state of Kansas. Python scripts were then utilized to export the GIS data into an input format for MLAEM, which implements the AEM mathematical representation presented earlier. Output from MLAEM were then imported to ArcGIS, which provides visualization and spatial analysis capabilities.

5 Conclusions

The Analytic Element Method provides a means to solve a set of governing equations - (2) and (3) - in an infinite domain. This technique is based upon formulating analytic elements that represent flow associated with points - (4), (5a) or (5b) - lines - (6), (7a) or (7b) - and polygons - (8). A means of easily obtain closed-form solutions for line integrals is also presented in (14). Each of these analytic element may be utilized in modeling specific types of hydrological feature, as shown in table 1.

An AEM geodatabase model is presented in figure 1 that organizes hydrogeologic data in a format directly related to the AEM representation of groundwater flow. This geodatabase extends previous efforts in the groundwater field using finite numerical techniques to vector based mathematical approaches. An example is presented that illustrates how the AEM geodatabase may be used to create and visualize groundwater models using MLAEM (Multi-Layer Analytic Element Model).

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