Evaluation of the European stock option by using the RBF approximation

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Abstract

This paper describes the application of a radial basis function approximation for evaluating European and American stock options. The European stock option is evaluated with the Black-Sholes equation and the boundary condition related to the exercise price. The Black-Sholes equation is solved with the Crank-Nicholson scheme and the radial basis function approximation. Numerical results are compared with the theoretical solutions to discuss the validity of the formulation. Finally, the extension of the present formulation for evaluating the American option is also explained.

1 Introduction

Recently, the financial derivatives are widely dealt and the importance is expanded. The importance of the derivative transaction is increasing for the adequate sharing of the financial risk. The option transaction is one of the most important financial derivatives and therefore, several schemes have been presented by many researchers for the pricing of the options [1, 2].

Several types of the financial options have been developed; European option, American option, Look-Back option, Exotic option and so on. In this study, we will consider the pricing of the European and the American options. The price of the European option can be evaluated as the solution of the Black-Sholes differential equation by taking the payoff condition in maturity day. The Black-Sholes equation is discretized according to the Crank-Nicholson scheme on the time axis and the option price is approximated with Radial Bases Function with unknown parameters at each time step. The initial values of the parameters are determined from the payoff condition in maturity day. Then, the parameters at the pricing day
are evaluated according to the backward algorithm from the maturity day to the pricing day. The numerical solutions are compared with the analytical ones.

Next, the present formulation will be extended for the pricing of the American option. The American option price is governed with the inequality related to Black-Scholes equation for the European option. Therefore, it can not be solved analytically. In this paper, the solution procedure for the European option is extended for the American option and the numerical solutions are compared with the solutions with finite differential method.

This paper is organized as follows. In the section 2, the evaluation scheme for the European option is explained and the numerical solutions are compared with the theoretical ones in the section 3. In the section 4, the evaluation scheme for the American option is explained and the numerical solutions are compared with the finite difference solutions in the section 5. The conclusions are discussed in the section 6.

2 Evaluation of European option

2.1 Black-Sholes equation [1, 2]

The stock price $S$, in the case of delivery of dividends, can be evaluated with probabilistic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dW$$

where $\mu$ and $\sigma$ denote the drift and the volatility, respectively. Now, we shall consider they are independent of the time $t$. Besides, $dW$ denotes the standard Wiener process.

Since the uncertainty of the derivative price $V = V(S, t)$ for the stock $S$ are equal to that of the stock $S$, the derivative $V$ is governance with the stochastic differential equation of the drift $m$ as follows.

$$\frac{dV}{V} = mdt + vdW$$

Now we assume that the resources are sank into the stock $S$ and its derivative $V$.

$$\frac{dP}{P} = x\frac{dS}{S} + (1-x)\frac{dV}{V}$$

where $x$ denotes the ratio of the resource sunk into the stock, which is taken from 0 to 1. From equations (1) and (2), we have

$$\frac{dP}{P} = x[\mu dt + \sigma dW]$$

$$+ (1-x)[mdt + vdW]$$

$$= [x\mu +(1-x)m]dt$$
\[ + [x\sigma + (1 - x)v]dW \]  

(4)

When the uncertainty of the portfolio can be neglected, the second term of the right-hand term in equation (4) should be zero. So, we have

\[ x\sigma + (1 - x)v = 0 \]

\[ x = \frac{v}{v - \sigma} \]  

(5)

From no arbitrage opportunity condition, the return of the risk-free resources should be equal to the domestic risk-free rate \( r_d \). The coefficient at the first term of the right-hand side in equation (4) is equal to \( r_d \). So, we have

\[ x\mu + (1 - x)m = r_d \]  

(6)

Substituting the equation (5) into the above equation, we have

\[ x\mu + (1 - x)m = \frac{v\mu - m\sigma}{v - \sigma} \equiv r_d \]  

(7)

Finally, we have the no arbitrage transaction condition as follows.

\[ \frac{\mu - r_d}{\sigma} = \frac{m - r_d}{v} \]  

(8)

Applying the Ito’s lemma to the derivative \( V \), we have

\[
dV = \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} \right) dt + \sigma S \frac{\partial V}{\partial S} dW \]  

(9)

Comparing the above equation with the equation (2), we have

\[ m = \frac{1}{V} \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} \right) \]  

(10)

\[ v = \frac{\sigma S}{V} \frac{\partial V}{\partial S} \]  

(11)

Substituting the equations (10) and (11) into the equation (8), we have

\[
\frac{\partial}{\partial t} V(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial t^2} V(S, t) + r \frac{\partial}{\partial S} V(S, t) - rV(S, t) = 0 \]

(12)

Besides, the boundary condition for the European put-option is given as

\[ V = \max(E - S, 0) \]  

where \( E \) denotes the exercise price for the option and the function \( \max(E - S, 0) \) estimates the larger value.
2.2 Solution procedure using RBF approximation

Discretizing the equation (12) with Crank-Nicholson Scheme, we have

\[
\frac{V(t + \Delta t) - V(t)}{\Delta t} + (1 - \theta) \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial t^2} + r \frac{\partial}{\partial S} - r \right) V(t + \Delta t) \\
+ \theta \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial t^2} + r \frac{\partial}{\partial S} - r \right) V(t) = 0
\]

(14)

where the parameter \( \theta \) is taken in the range of \( 0 \leq \theta \leq 1 \).

Defining the parameters \( V(t) = V^n \) and \( V(t + \Delta t) = V^{n+1} \), we have

\[ HV^{n+1} = GV^n \]

(15)

\( H \) and \( G \) are the differential operators defined as

\[ H = 1 + \alpha \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial t^2} + r \frac{\partial}{\partial S} - r \right) \]

\[ G = 1 - \beta \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial t^2} + r \frac{\partial}{\partial S} - r \right) \]

where

\[ \alpha = (1 - \theta)\Delta t \]

\[ \beta = \theta \Delta t \]

In this study, the following radial bases function is adopted:

\[ \phi(S, S_j) = \sqrt{1 + r^2_j} \]

(16)

where \( r^2_j = \| S - S_j \| \).

The derivative price \( V \) governed with the equation (12) is approximated with the RBF function as follows.

\[ V = \sum_{j=1}^{N} \lambda_j \phi_j \]

(17)

where \( N \) and \( \lambda_j \) denote the total number of data points and the unknown parameters, respectively.

Substituting equation (17) into (23), we have

\[ \sum_{j=1}^{N} H \lambda_j^{n+1} \phi_j = \sum_{j=1}^{N} G \lambda_j^n \phi_j \]

(18)
Table 1: Simulation parameters.

<table>
<thead>
<tr>
<th>Exercise price</th>
<th>$E = 10.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest rate</td>
<td>$r = 0.05$</td>
</tr>
<tr>
<td>Volatility</td>
<td>$\sigma = 0.2$</td>
</tr>
<tr>
<td>Remaining period</td>
<td>$T = 0.5$ Year</td>
</tr>
<tr>
<td>Number of time-steps</td>
<td>100</td>
</tr>
<tr>
<td>Number of approximating points</td>
<td>$N = 121$</td>
</tr>
<tr>
<td>Parameter for Crank-Nicholson Scheme</td>
<td>$\theta = 0.5$</td>
</tr>
</tbody>
</table>

Table 2: Numerical results.

<table>
<thead>
<tr>
<th>Stock $S$</th>
<th>Theoretical</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>9.7531</td>
<td>9.7531</td>
</tr>
<tr>
<td>2.0</td>
<td>7.7531</td>
<td>7.7531</td>
</tr>
<tr>
<td>4.0</td>
<td>5.7531</td>
<td>5.7531</td>
</tr>
<tr>
<td>6.0</td>
<td>3.75318</td>
<td>3.75318</td>
</tr>
<tr>
<td>8.0</td>
<td>1.79871</td>
<td>1.79823</td>
</tr>
<tr>
<td>10.0</td>
<td>0.441972</td>
<td>0.440547</td>
</tr>
<tr>
<td>12.0</td>
<td>0.0483444</td>
<td>0.0478026</td>
</tr>
<tr>
<td>14.0</td>
<td>0.00277485</td>
<td>0.00270994</td>
</tr>
<tr>
<td>16.0</td>
<td>0.000103001</td>
<td>0.000100917</td>
</tr>
</tbody>
</table>

2.3 Solution algorithm

The algorithm of the present scheme is as follows.

1. Discretize from $t = 0$ to $t = T$ with time-step $\Delta t$.
2. Specify $V(S, T)$ at the time $T$ from the boundary condition and calculate $\lambda_j^T$ at the time $T$.
3. Calculate $\lambda_j^{T-\Delta t}$ from equation (18).
4. Repeat step 2 and 3 till $t = 0$.
5. Evaluate $V(S, 0)$ from $\lambda_j^0$. 
3 Numerical example for European option

We shall consider the pricing of the European put-option of which parameters are given as shown in Fig. 1.

The numerical solutions are compared with the theoretical ones in Table 2. We notice that the numerical solutions well agree with the theoretical ones.

4 Evaluation of American option

4.1 Governing equation for American option [1, 2]

The American option $V(S, t)$ for the stock price $S$ is governed with the inequality:

$$
\frac{\partial}{\partial t} V(S, t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial t^2} V(S, t) \\
+ r \frac{\partial}{\partial S} V(S, t) - r V(S, t) \leq 0
$$

(19)

$$
V(S, t) \geq h
$$

(20)

where $h$ is payoff function, which is defined as

$$
h = \begin{cases} 
\max(E - S, 0) & \text{for Put} \\
\max(S - E, 0) & \text{for Call}
\end{cases}
$$

(21)

where $E$ denotes the exercise price for the option and the function $\max(E = S, 0)$ estimates the larger value.

4.2 Solution procedure using RBF approximation

The use of the Crank-Nicholson scheme derives the following equations from equation (19).

$$
\frac{V(t + \Delta t) - V(t)}{\Delta t} \\
+ (1 - \theta) \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial t^2} + r \frac{\partial}{\partial S} - r \right) V(t + \Delta t) \\
+ \theta \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial t^2} + r \frac{\partial}{\partial S} - r \right) V(t) \leq 0
$$

(22)

Defining the parameters $V(t) = V^n$ and $V(t + \Delta t) = V^{n+1}$, we have

$$
HV^{n+1} \leq GV^n
$$

(23)

In this study, the following radial bases function is adopted:

$$
\phi(S, S_j) = \sqrt{1 + r_j^2}
$$

(24)

where $r_j^2 = \|S - S_j\|$.
The derivative price $V$ governed with the equation (12) is approximated with the RBF function as follows.

$$V = \sum_{j=1}^{N} \lambda_j \phi_j$$  \hspace{1cm} (25)

where $N$ and $\lambda_j$ denote the total number of data points and the unknown parameters, respectively.

Substituting equation (25) into (23), we have

$$\sum_{j=1}^{N} H \lambda_j^{n+1} \phi_j \leq \sum_{j=1}^{N} G \lambda_j^n \phi_j$$  \hspace{1cm} (26)

### 4.3 Solution algorithm

The American option price is governed with the inequality. In the solution algorithm, the tentative option price is evaluated from the equality. Then, the price is updated so as to satisfy the inequality.

The algorithm of the present scheme is as follows.

1. Discretize from $t = 0$ to $t = T$ with time-step $\Delta t$.
2. Specify $V(S, T)$ at the time $T$ from the boundary condition and calculate $\lambda_j^T$ at the time $T$.
3. Calculate $\lambda_j^{T-\Delta t}$ from equation (26).
4. Evaluate $V(S, t)$.
5. If $V(S, t) < h$, $V(S, t) \leftarrow h$.
6. Repeat step 3 and 4 till $t = 0$.
7. Evaluate $V(S, 0)$ from $\lambda_j^0$.

### 5 Numerical example for American option

We shall consider the pricing of the American put-option of which parameters are given as shown in Table 1. The numerical solutions are compared with the finite difference solutions in Table 3. We notice that the numerical solutions well agree with the finite difference ones.

### 6 Conclusions

This paper describes the numerical solution scheme for the evaluation of the European option using Radial Bases Function approximation. In the formulation, the Black-Sholes equation is expanded with Crank-Nicholson scheme and the stock option price is approximated with RBF functions. The present scheme is applied to the evaluation of the European stock option. The numerical solutions well agree with the theoretical ones. So, after that, the formulation was extended for the evaluating the American option. The numerical results well agreed with the solutions.
Table 3: Numerical results.

<table>
<thead>
<tr>
<th>Stock S</th>
<th>Finite Difference</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>10.0</td>
<td>10.0</td>
</tr>
<tr>
<td>2.0</td>
<td>7.9975</td>
<td>8.0</td>
</tr>
<tr>
<td>4.0</td>
<td>5.9975</td>
<td>6.0</td>
</tr>
<tr>
<td>6.0</td>
<td>3.9975</td>
<td>4.0</td>
</tr>
<tr>
<td>8.0</td>
<td>1.99752</td>
<td>2.0</td>
</tr>
<tr>
<td>10.0</td>
<td>0.463555</td>
<td>0.464551</td>
</tr>
<tr>
<td>12.0</td>
<td>0.0501426</td>
<td>0.0494443</td>
</tr>
<tr>
<td>14.0</td>
<td>0.00309405</td>
<td>0.00281253</td>
</tr>
<tr>
<td>16.0</td>
<td>0.00013847</td>
<td>0.000109521</td>
</tr>
</tbody>
</table>

with finite difference method. In the future, we are going to extend the present formulation for the evaluating the other options such as Look-Back and Asian options.

References
