Direct solution of certain ill-posed boundary value problems by the collocation method

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Abstract

Numerical solution of ill-posed boundary value problems normally requires the conversion of the problems into well-posed ones and their solution by an iterative procedure. This paper offers a direct solution procedure to some ill-posed problems by utilizing the collocation method.

*Keywords:* ill-posed problem, inverse problem, Cauchy problem, radial basis function, collocation method, meshless method.

1 Introduction

Direct mathematical solution of engineering problems requires the problems to be set up as well-posed boundary value problems. Engineering problems, however, are not always set up as well-posed problems. Due to the issues of accessibility and cost of measurement, data can be missing on parts of the boundary. In other situations, the location of a boundary is itself not known. To make up for the missing information, a part of the boundary may be over-specified, or potential values can be given in interior points.

There are many problems of these types. For example, in groundwater flow, boundary conditions are found along geological features that may not be connected to form a closed boundary. On the other hand, there often exist monitoring wells scattered in the domain of interest to provide the observation of piezometric head. In geophysical prospecting, underground features are inaccessible, but the surface location allows the imposition and measurement of electrical potential and flux such that both the Dirichlet and the Neumann conditions are provided. The same situation exists in the industrial application of
A typical way to solve ill-posed problems is to first convert them into well-posed ones. On the part of the boundary where boundary data are missing, trial values are assigned. On other parts of the boundary, or in the interior of the domain, redundant conditions are ignored. With this newly formulated well-posed boundary value problem, a standard numerical technique, such as the finite element method or the finite difference method, is used for the solution. Once the solution is obtained, it is checked against the redundant data for consistency. If the solution does not match, the assumed boundary conditions need to be adjusted. Optimization schemes are used to find the direction of adjustment that leads to a converged solution. Depending on the nature of the problem and the robustness of the optimization scheme, this search process may or may not be efficient, and may or may not even be stable. Hence solving ill-posed problems can be a tricky business.

In this article we present a different approach for solving ill-posed boundary value problems. By utilizing the collocation method, we demonstrate that certain ill-posed problems can be solved in a single step, without trials and iterations.

2 Ill-posed problems

We shall use the solution of Poisson equation as examples. Given the potential $u$ satisfying the Poisson equation

$$\nabla^2 u = f(x) \quad \text{in} \ \Omega$$

and the boundary conditions

$$u = g_1(x) \quad \text{on} \ \Gamma_D$$
$$\frac{\partial u}{\partial n} = g_2(x) \quad \text{on} \ \Gamma_N$$

(2)

where $\Omega$ is the solution domain with the boundary $\Gamma$, and $\Gamma_D$ and $\Gamma_N$ are the boundary with Dirichlet and Neumann type boundary conditions, respectively, a well-posed boundary value problem requires $\Gamma_D \cup \Gamma_N = \Gamma$, $\Gamma_D \cap \Gamma_N = \emptyset$, and $\Gamma_D \neq \emptyset$. In other words, on each part of the boundary there is one and only one boundary condition.

On the other hand, if the boundary conditions are specified such that $\Gamma_D \cup \Gamma_N \neq \Gamma$ or $\Gamma_D \cap \Gamma_N \neq \emptyset$, and in addition, there exist the internal conditions

$$u(x_j) = \bar{u}_j; \quad j = 1, \ldots, n_i; \quad x_j \in \Omega$$

(3)

where $n_i$ is the number of internal nodes, then the problem is ill-posed. In other words, the boundary conditions are such that it can be missing on certain part of
the boundary, and/or over-specified (both the Dirichlet and Neumann conditions exist) on another part. There can also exist conditions on internal points.

3 Direct solution of ill-posed problems

We assume that the approximate solution \( \hat{u} \) of the above system of equations is given by the series

\[
\hat{u}(x) = \sum_{i=1}^{n} \alpha_i \varphi_i(x)
\]

in which \( \varphi_i(x) \) are basis functions, and \( \alpha_i \) are constant coefficients to be determined. To ensure that the governing equation (1) is satisfied, we collocate the approximate solution \( \hat{u} \) on a set of \( n_i \) interior and boundary points \( x_j \), such that

\[
\sum_{i=1}^{n} \alpha_i \nabla^2 \varphi_i(x_j) = f(x_j); \quad j = 1, \ldots, n_i; \quad x_j \in (\Omega \cup \Gamma)
\]

To satisfy the boundary conditions (2), we require

\[
\sum_{i=1}^{n} \alpha_i \varphi_i(x_j) = g_1(x_j); \quad j = n_1 + 1, \ldots, n_1 + n_2; \quad x_j \in \Gamma_D
\]

\[
\sum_{i=1}^{n} \alpha_i \frac{\partial \varphi_i(x_j)}{\partial n} = g_2(x_j); \quad j = n_1 + n_2 + 1, \ldots, n_1 + n_2 + n_3; \quad x_j \in \Gamma_N
\]

Further, the interior condition (3) needs to be satisfied

\[
u(x_j) = \tilde{u}_j; \quad j = n_1 + n_2 + n_3 + 1, \ldots, n_1 + n_2 + n_3 + n_4; \quad x_j \in \Omega
\]

where \( n_1 + n_2 + n_3 + n_4 = n \). Equations (5) to (7) then form a system of \( n \) linear equations, which can be solved for the \( n \) unknown coefficients \( \alpha_i \). The full solution is then defined at all points in \( \Omega \) as given by (4). This solution procedure is known as the collocation method.

We notice that in the collocation method there is no difference between the solution procedure for the well-posed and the ill-posed problems. For both problems, the governing equation is enforced on a number of points in \( \Omega \). The boundary and the internal conditions are also enforced wherever they are available. Both types of problems are solved in a single step without iteration.

There is a range of choices for \( \varphi_i \). If certain orthogonal basis is chosen, such as the Chebyshev polynomials or the Fourier series, the method is known as the orthogonal collocation method. The error convergence of the orthogonal series is generally superior. These methods however apply only to box geometry; hence are not suitable for general engineering applications.
If we choose $\phi_i$ as the radial basis functions, $\phi_i = \phi_i(r)$, where $r$ is the Euclidean distance between two points, the method becomes the Kansa method [1]. The use of radial basis function allows the collocation points to be scattered over the domain and is well suited for problems of arbitrary geometry. Particularly, the use of the inverse multiquadric as the basis function

$$\phi(r) = \left(r^2 + c^2\right)^{-1/2}$$

where $c$ is a constant referred to as the shape parameter, leads to a method with exponential convergence properties [2]. The multiquadric collocation method is the method of choice of the present work.

It is of interest to note that if we choose $\phi_i$ to be harmonic functions, namely functions that satisfy the Laplace equation, then the approximate solution (4) automatically satisfies the governing equation. In that case, collocation inside the domain given by (5) is no longer needed. Only (6) and (7) will be executed. This is known as the Trefftz method [3]. Furthermore, if $\phi_i$ is chosen to be the fundamental solution (free-space Green’s function) of the governing equation with the singularity placed outside of the solution domain, the method becomes the method of fundamental solutions [4].

4 Numerical examples

4.1 Problems with missing boundary condition and internal data

The following ill-posed problem of potential flow was investigated by Onishi [5]. On a square domain $[0,3] \times [0,3]$, Dirichlet conditions are prescribed on three sides of the boundary:

$$u(x,3) = x^2 - 9; \quad u(0,y) = -y^2; \quad u(3,y) = 9 - y^2$$

No boundary condition is given on the fourth side $\{y = 0; 0 \leq x \leq 3\}$. In the interior, the potential values are known on four points: $u(1,1) = 0, \quad u(2,1) = 3, \quad u(1,2) = -3, \quad \text{and} \quad u(2,2) = 0$. These boundary conditions and internal values are based on the exact solution of Laplace equation

$$u(x,y) = x^2 - y^2$$

Onishi [5] solved this problem using a FEM. In each step, a well-posed boundary value problem is solved using assumed boundary condition on the side with missing data. The residuals at the four internal points are checked and are minimized using an optimization procedure. The solution at the point $(1.5, 0)$, which is the mid-point of the boundary with missing data, using two different meshes is compared with the exact solution as shown in Table 1.
Table 1: Comparison of error of potential at the point (1.5, 0).

<table>
<thead>
<tr>
<th>Method</th>
<th>Potential value</th>
<th>Percent error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact solution</td>
<td>2.250</td>
<td>–</td>
</tr>
<tr>
<td>Onishi, FEM 36 elements</td>
<td>2.323</td>
<td>3.2</td>
</tr>
<tr>
<td>Onishi, FEM 144 elements</td>
<td>2.341</td>
<td>4.0</td>
</tr>
<tr>
<td>RBF, 49 collocation nodes (c = 3)</td>
<td>2.296</td>
<td>2.0</td>
</tr>
<tr>
<td>RBF, 49 collocation nodes (c = 4)</td>
<td>2.251</td>
<td>0.04</td>
</tr>
</tbody>
</table>

In the present RBF collocation solution, the collocation points form a 7×7 grid as shown in Figure 1. Two types of collocation points are used: on the Dirichlet boundary and the four interior points (shown in ● symbol), the potential values are collocated; and on the remainder of the interior grid points and the unknown boundary (shown in ×), the governing equation is collocated.

![Figure 1: RBF collocation nodes for example 1. (●: collocation nodes for potential; ×: collocation nodes for governing equation.).](image)

Table 1 compares the potential value of the current RBF solution with the FEM solution of Onishi [5] at the point (1.5, 0). We observe that with this relatively sparse mesh (total 49 degrees of freedom) and the use of \( c = 3 \), the accuracy is somewhat better than the FEM solution. As demonstrated in Cheng, et al. [2], the multiquadric collocation method has the exponential convergence property, which can be achieved either by refining the mesh or by increasing the
value of $c$ value. In the current problem, we simply increase the $c$ value from 3 to 4, without refining the mesh, the error reduces to 0.04%, as shown in Table 1. In making comparison in Table 1, we should emphasize that the Onishi FEM solution required many iterations to reach the final result. The current solution is obtained in a single step.

4.2 Problems with over-specified boundary condition

These ill-posed problems of steady state heat conduction were considered by Lesnic [6]. Figure 2 shows the geometry of a square domain $\Omega = [0,1] \times [0,1]$. Four different configurations of boundary conditions are presented. The boundary condition is missing on either one side or two sides of the domain. To make up the shortage of information, over-specified (Cauchy) conditions are prescribed on one side or two sides as shown. The boundary conditions, wherever they exist, are defined based on the exact solution

$$T(x,y) = \cos(x) \cosh(y) + \sin(x) \sinh(y)$$


Figure 2: Four cases of Cauchy problems for steady state heat conduction with different boundary conditions.

The performance of the Lesnic solution is summarized in Table 2. The number of boundary elements used ranges from 40 to 160, which means 10 to 40
elements per side. The percentage error at the mid-point of the left side boundary, \((0, 0.5)\), is presented. We observe that for cases 1, 2 and 3, the accuracy for the predicted temperature is excellent, with errors less than 1%. The prediction of normal heat flux at that point is not as good, but is reasonable, with maximum error of 6% for case 2. To obtain these solutions, 100 to 1,000 iterations were needed. Case 4 is more difficult. Boundary conditions are missing on two sides of the boundary. On the remaining two sides, Cauchy conditions are prescribed. In this case, large errors in temperature (13%) and heat flux (50%) are observed after 10,000 iterations, indicating that the solution is unstable.

Table 2: Percentage error of Lesnic’s iterative BEM solution at the middle point of left side boundary, \((0, 0.5)\).

<table>
<thead>
<tr>
<th></th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of elements</td>
<td>40</td>
<td>160</td>
<td>160</td>
<td>160</td>
</tr>
<tr>
<td>Number of iterations</td>
<td>100</td>
<td>1000</td>
<td>1000</td>
<td>10000</td>
</tr>
<tr>
<td>Error in temperature (%)</td>
<td>0.4</td>
<td>0.5</td>
<td>0.3</td>
<td>13</td>
</tr>
<tr>
<td>Error in heat flux (%)</td>
<td>2</td>
<td>6</td>
<td>1.5</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 3: Percentage error of RBF collocation solution for temperature at the middle point of left side boundary, \((0, 0.5)\), for different grids.

<table>
<thead>
<tr>
<th>Grid</th>
<th>Case 1 (%)</th>
<th>Case 2 (%)</th>
<th>Case 3 (%)</th>
<th>Case 4 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6×6</td>
<td>0.5</td>
<td>0.6</td>
<td>1.7</td>
<td>0.4</td>
</tr>
<tr>
<td>8×8</td>
<td>0.1</td>
<td>0.009</td>
<td>0.009</td>
<td>0.06</td>
</tr>
<tr>
<td>10×10</td>
<td>0.009</td>
<td>0.000</td>
<td>0.009</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Table 4: Heat flux percentage error of RBF collocation solution at the middle point of left side boundary, \((0, 0.5)\), for different grids.

<table>
<thead>
<tr>
<th>Grid</th>
<th>Case 1 (%)</th>
<th>Case 2 (%)</th>
<th>Case 3 (%)</th>
<th>Case 4 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6×6</td>
<td>3.2</td>
<td>0.2</td>
<td>3.4</td>
<td>4.0</td>
</tr>
<tr>
<td>8×8</td>
<td>0.6</td>
<td>0.06</td>
<td>0.06</td>
<td>0.06</td>
</tr>
<tr>
<td>10×10</td>
<td>0.1</td>
<td>0.02</td>
<td>0.2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

These problems are also solved using the RBF collocation method. Several grid systems, ranging from \(6\times6\) to \(10\times10\), are used. When Cauchy condition exists, both conditions are collocated with collocation nodes slightly separated from each other. The value \(\epsilon = 4\) is used for the multiquadric. In Table 3, the errors of temperature at the mid-point of the left boundary, \((0, 0.5)\), are displayed. We observe that the accuracy of solution is superior, with error converges to a tiny fraction of a percentage point for all four cases. Even for case 4, where the Lesnic [5] solution failed even after 10,000 iterations, the current
result is excellent with only one iteration. The errors in heat flux normal to the boundary at \((0, 0.5)\) are reported in Table 4. These errors are generally larger than the temperature predictions as shown in Table 3. However, the accuracy is still excellent, with errors converged to a fraction of a percentage point.

4.3 Shape identification problems

Hon and Wu [7] investigated the following Cauchy problem for shape identification, which can arise in the industrial application of non-destructive testing. For a half plane, \(y \geq 0\), the Cauchy boundary condition is prescribed on the segment \(\{-0.5 \leq x \leq 0.5; y = 0\}\), where

\[
\begin{align*}
  u &= f(x) = x^2 - 0.31x - 1.089 \\
  \frac{\partial u}{\partial y} &= g(x) = 0.1x + 2.19
\end{align*}
\]

(12)

The task is to locate the internal shape along which the potential value \(u = 0\) is satisfied. The exact solution of this problem is

\[
  u = (x - 0.1)^2 - (y - 1.1)^2 + 0.1(x - 0.1)(y - 1.1) + 0.1
\]

(13)

For the RBF collocation solution, we collocate for the Dirichlet and the Neumann conditions each at 5 points on the Cauchy boundary. In the domain, the governing equation is collocated on a \(9 \times 9\) grid. The shape parameter used is \(c = 3\). After the solution, the zero potential trajectories are searched. The result is plotted in Figure 3. The numerical solution is shown in symbols, which can be compared with the exact solution plotted in solid lines. We observe that the numerical solution captured the lower branch very well. For the upper branch, we find a slight deviation. These results are comparable with Hon and Wu’s solution [7].

![Figure 3: Shape identification problem: Solution with exact Cauchy data.](image-url)
Since Cauchy problem is known to be unstable, Hon and Wu [7] further conducted numerical experiment to test the sensitivity of solution subject to noisy boundary data. Noise levels from $10^{-5}$ to $10^{-4}$ were tested. According to Hon’s findings, when the noise reached $10^{-4}$ for both the potential and the flux, the shapes became unidentifiable, and the solution failed. Since this noise level is rather small, it is of interest to re-examine the numerical experiment.

In our experiment, a much larger noise range, from $k = 10^{-5}$ to $10^{-2}$, are tested. The noise is imposed in the following fashion: for each input data of Dirichlet and Neumann condition, its values is modified to become

$$u = (1 + k\varepsilon) f(x)$$
$$\frac{\partial u}{\partial y} = (1 + k\varepsilon) g(x)$$

(14)

where $f(x)$ and $g(x)$ are boundary conditions defined in (12), $\varepsilon$ is a uniformly distributed random number between $[-1, +1]$, and $k$ is the amplitude of noise. The numerical results are shown in Figure 4.

![Figure 4: Shape identification problem: Solution with perturbed data.](image)

In Figure 4 we observe that for noise up to the level $k = 10^{-4}$, the solution is not sensitive to perturbation, which is different from what Hon and Wu [7] had reported. Solutions for the three cases $k = 0$, $10^{-5}$, and $10^{-4}$ are practically identical to each other and are plotted on the same curve for the upper and lower branches. For noise level up to $10^{-3}$, however, the solution for the upper branch starts to deviate, while that for the lower branch remains unchanged. When $k$ is further increased to $10^{-2}$, we observe that even the solution of the lower branch starts to deteriorate, but the shape is still identifiable. For the upper branch, the
solution for $k = 10^{-2}$ breaks down and the predicted shape bears no resemblance to the true one; hence it is not plotted in Figure 4.

5 Conclusions

In this paper we presented an efficient numerical technique for solving several types of ill-posed boundary value problems with practical applications in mind. The rationale for developing this technique is summarized as follows:

1. The RBF collocation method is a popular method for solving well-posed boundary value problems because of its “meshless” nature.
2. Among the RBFs, the multiquadric functions possess the exponential convergence rate in error estimate, which is highly efficient.
3. The RBF collocation method applies to well-posed boundary value problems as well as ill-posed problems using the same solution procedure.
4. The solution procedure involves a single-step, which is much more efficient than the conventional iterative techniques for solving ill-posed problems.
5. Ill-posed problems can be unstable from their mathematical nature. Iterative solution methods normally encounter difficulty in solution convergence. This difficulty is circumvented in this direct solution procedure.

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References


