

# Extension of CVBEM to dispersive flow in the presence of distributed fractures

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## Abstract

This study aims to extend the applicability of the complex variable boundary element method (CVBEM) to convective-dispersive flow in the presence of distributed fractures. It is difficult to model the behavior of detailed flow around thin objects such as fractures, owing to singularity at the object edges. In addition, when tracking streamlines that pass through fractures, one cannot identify the outward streamline corresponding to a specific inward streamline. This article shows how complex variables can be used to remedy these difficulties. It then proposes a numerical scheme for modeling potential flow in the presence of distributed fractures. The singular solution for flow around fractures is coupled with the CVBEM through singularity programming. Streamlines can be tracked through fractures by using the stream function.

Another research target is the effect of dispersivity on flow behavior. In order to utilize the analytical solution of convective-dispersive flow obtained for one-dimensional (1D) problems, two-dimensional (2D) flow is decomposed into flow along multiple 1D streamlines. Variations in velocity along the streamlines can be taken into account with the aid of a set of coordinate transformations, which the CVBEM accurately evaluates. The complete 2D solution is recovered by combining the individual 1D solutions.

The numerical results of three example problems: convective flow through a single fracture, flow with various dispersivities, and dispersive flow through multiple fractures, confirm that the developed model enables accurate flow computations with dispersivity in the presence of distributed fractures.

*Keywords: complex variable, BEM, fluid flow, Laplace, dispersion, streamline, tracer, fracture, singularity, conformal mapping.*



## 1 Introduction

The applicability of the boundary element method (BEM) covers a wide range of flow problems, including homogeneous and sectionally homogeneous media [1], functional heterogeneity [2], random heterogeneity [3], moving-boundary problems [4], and stochastic groundwater problems [5]. Although a variety of problems have been investigated, many flow problems remain to be solved.

For instance, thin objects are hard to model in a real plane, due to singular behavior around the object edges. This difficulty can be overcome by conformally mapping the thin object into some other plane where a singular solution for the object is readily obtained. Complex variables are suitable for such mapping; hence, it is interesting to apply the complex variable BEM (CVBEM) [6] to try to model the thin objects [7, 8]. Another difficulty regarding flow around thin objects is in tracking streamlines. The simplest tracking method is that of the velocity vector [9]. However, this method cannot track streamlines that pass through thin objects; once a streamline flows into a thin object, one cannot identify the outward streamline corresponding to the inward streamline. This difficulty, too, could be overcome by using the CVBEM, which provides both the velocity potential and the stream function. Since the streamlines coincide with the equi-stream-function lines, the streamlines that pass through thin objects could be tracked by evaluating the stream functions [8, 10].

The CVBEM exhibits two potential features: thin-object modeling and streamline tracking. This prompts us to establish and examine a numerical scheme to track streamlines through fractures. In addition, hydrodynamic dispersion affects the flow behavior. The purpose of this study is to extend the applicability of the CVBEM to dispersive flow in the presence of distributed fractures.

## 2 Flow physics

For an incompressible fluid of viscosity  $\mu$  and pressure  $p$  irrotationally flowing through a homogeneous medium of permeability  $k$ , the velocity potential  $\Phi = (k/\mu)p$  satisfies the Laplace equation:  $\nabla^2\Phi = 0$ . The harmonic conjugate of  $\Phi$  is the stream function  $\Psi$ , and the Laplace equation in terms of the complex velocity potential  $\Omega$  stands as

$$\nabla^2\Omega = \nabla^2(\Phi + i\Psi) = 0. \quad (1)$$

### 2.1 Sink/source singularity

The analytic solution  $\Omega_{wj}$  for the  $j$ -th sink can be written as [8]

$$\Omega_{wj}(z) = \frac{q_{wj}}{2\pi h} \ln(z - z_{wj}), \quad (2)$$

where  $q_{wj}$  is the strength,  $h$  the medium thickness, and  $z_{wj}$  the sink location. With a negative value of  $q_{wj}$ , eqn (2) gives the solution for the sources.



## 2.2 Fracture singularity

A fracture  $z_1 z_2$  of permeability  $k_f$  and aperture  $w_f$  is considered, as shown in fig. 1. The difference in  $\Psi$  on either side (+ or -) of the fracture is equal to the flow rate per unit thickness inside the fracture:

$$\Psi^+ - \Psi^- = \frac{k_f w_f}{k} \frac{d\Phi}{ds}, \quad (3)$$

where  $s$  is the length measured along the fracture and Darcy's law is applied to the flow through the fracture. In contrast, the velocity potential on one side of the fracture must equal that on the other side:  $\Phi^+ - \Phi^- = 0$ . Thus, the singular solution  $\Omega_{fj}$  for the  $j$ -th fracture can be defined such that its real part  $\Phi_{fj}$  is continuous and its imaginary part  $\Psi_{fj}$  is discontinuous across the fracture.

In the conformally mapped  $\chi$  plane (fig. 1), these conditions can be met if the function  $\Omega_{fj}$  is chosen such that

$$\Omega_{fj}(\chi) = \overline{\Omega_{fj}(\bar{\chi})}. \quad (4)$$

The  $\chi$  plane is obtained through the Joukowski transformation:

$$\chi(w) = w + \sqrt{(w-1)(w+1)}, \quad (5)$$

where  $w(z) = [2z - (z_1 + z_2)] / (z_2 - z_1)$ .

A simple solution for eqn (4) is given as [11]

$$\Omega_{fj}(\chi) = \sum_{l=1}^{n_L} \frac{a_l}{\chi^l} + \sum_{l=1}^{n_p} \left( \frac{\beta_l \delta_l}{\chi - \delta_l} + \frac{\overline{\beta_l \delta_l}}{\chi - \bar{\delta}_l} \right), \quad (6)$$

where  $\beta_l = b_l + ic_l$  and  $a_l$ ,  $b_l$ , and  $c_l$  are real numbers. The first summation can be referred to as a central expansion of order  $n_L$  and the second as first-order expansions about  $n_p$  complex conjugate pairs of imaginary poles located at  $\delta_l$  and  $\bar{\delta}_l$ , shown in fig. 1.

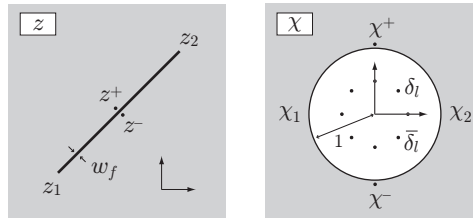


Figure 1: Joukowski transformation.

### 2.3 Flow with dispersion

Along a direction of irrotational flow, a convection-dispersion equation for incompressible fluids is written as [9]

$$\phi \frac{\partial C}{\partial t} + u \frac{\partial C}{\partial s} - \alpha |u| \frac{\partial^2 C}{\partial s^2} = 0, \quad (7)$$

where  $C$  is the concentration,  $s$  the arc length along the flow direction,  $\phi$  the porosity,  $u$  the superficial velocity, and  $\alpha$  the longitudinal dispersivity. Gelhar and Collins [12] utilized boundary layer techniques to convert eqn (7) into a simple differential equation through a set of coordinate transformations,  $\tau$  and  $\omega$ .  $\tau$  is the time of flight (TOF), which transforms spatial coordinates to the time required for the fluid particle to move from the origin to  $s$ , defined as

$$\tau(s) = \int_0^s \frac{\phi dr}{|u(r)|}. \quad (8)$$

The transformation  $\omega$  is defined as

$$\omega(\bar{s}) = \int_0^{\bar{s}} \frac{\phi^2 dr}{u^2(r)}, \quad (9)$$

where  $\bar{s}$  is the hypothetical front that would exist between a tracer and a native fluid if no dispersion occurred.

With  $\tau$  and  $\omega$ , eqn (7) can be reduced to the following equation:

$$\frac{\partial C}{\partial \omega} - \alpha \frac{\partial^2 C}{\partial \eta^2} = 0, \quad (10)$$

where  $\eta = \tau - t$ . Eqn (10) is valid when  $\sqrt{\alpha/L} \ll 1$ . Subject to the initial condition:  $C(s, 0) = 0$ , and to two boundary conditions:  $C(0, t) = C_0$  and  $C(s \rightarrow \infty, t) = 0$ , the solution of eqn (10) is obtained as [12]

$$C(\tau, t) = \frac{C_0}{2} \operatorname{erfc} \left( \frac{\tau - t}{\sqrt{4\alpha\omega}} \right). \quad (11)$$

## 3 CVBEM and streamline simulation

### 3.1 Formulations of CVBEM

Cauchy's integral formula applies to the complex velocity potential:

$$\Omega(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\Omega(\zeta)}{\zeta - z} d\zeta, \quad (12)$$

where  $z$  is the point of interest interior to the boundary  $\Gamma$  and  $\zeta$  is the point on  $\Gamma$ . The boundary  $\Gamma$  is discretized into  $n_b$  straight-line elements, and the boundary



values are linearly interpolated along the elements. The discretized expression for eqn (12) is obtained as [6, 8]

$$\Omega(z) = \frac{1}{2\pi i} \sum_{j=1}^{n_b} \frac{(z - \zeta_j)\Omega_{j+1} - (z - \zeta_{j+1})\Omega_j}{\zeta_{j+1} - \zeta_j} \ln \frac{\zeta_{j+1} - z}{\zeta_j - z}, \quad (13)$$

where  $\Omega_j$  is the complex velocity potential at the nodal point  $\zeta_j$ .

When sinks/sources and/or fractures are included in the flow problem, the corresponding singularities are readily coupled with the CVBEM by singularity programming [7, 8]. According to the principle of superposition, the complete solution  $\Omega$  is expressed as a summation of non-singular ( $\Omega^{ns}$ ) and singular ( $\Omega^s$ ) solutions:

$$\Omega = \Omega^{ns} + \Omega^s = \Omega^{ns} + \sum_{j=1}^{n_w} \Omega_{wj} + \sum_{j=1}^{n_f} \Omega_{fj}, \quad (14)$$

where  $n_w$  is the number of sinks/sources and  $n_f$  is the number of fractures.  $\Omega_{wj}$  and  $\Omega_{fj}$  are given by eqns (2) and (6), respectively. To determine the real constants  $a_l$  and the complex constants  $\beta_l$  in eqn (6), the flow condition, eqn (3), is applied at selected control points along the fracture.

### 3.2 Streamline tracking

Let  $z_j$  be a point on a certain streamline, which has a stream function of  $\Psi_0$ . Then, the Euler method gives the first-order estimate of the next point ( $z_{j+1/2}$ ) on  $\Psi_0$  as

$$z_{j+1/2} = z_j + \frac{\overline{w_j}}{|w_j|} \Delta s, \quad (15)$$

where  $\Delta s$  is some fixed length and  $\overline{w_j}$  is the complex conjugate of the complex velocity  $w_j$  evaluated at  $z_j$ . To improve the accuracy, the second approximation can be obtained by the Newton-Raphson procedure along the equi-potential that passes through  $z_{j+1/2}$  [11]:

$$z_{j+1} = z_{j+1/2} + i \frac{\Psi_{j+1/2} - \Psi_0}{w_{j+1/2}}. \quad (16)$$

Eqn (16) is recursively applied to obtain  $z_{j+1}$  until  $|\Psi_{j+1/2} - \Psi_0|$  is less than a tolerance value.

The superficial time  $\Delta t_j$  required for the fluid particle to move from  $z_j$  to  $z_{j+1}$  is estimated with

$$\Delta t_j = \frac{2|z_{j+1} - z_j|}{|w_j + w_{j+1}|} = \frac{\Delta r_j}{|u_j|}, \quad (17)$$

where  $u_j$  is the average superficial velocity from  $z_j$  to  $z_{j+1}$  and  $\Delta r_j$  is the distance between  $z_j$  and  $z_{j+1}$  [8, 10].

When there exist fractures in the flow domain, some of the streamlines may flow into and out of the fracture as shown in fig. 2 (left). Fig. 2 (right) shows the



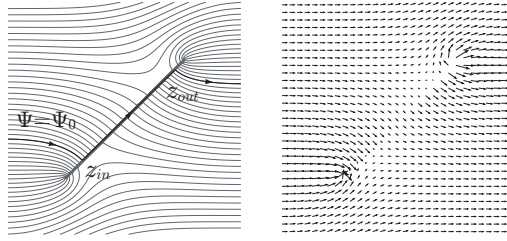


Figure 2: Streamlines and the corresponding velocity-vector field.

corresponding velocity-vector field. The velocity vectors are discontinuous around the fracture and cannot be used to identify an outflux point. The stream function is used to overcome this difficulty [8].

To check whether or not a certain streamline flows into the fracture, the maximum and minimum values of the stream function along the fracture are first computed. If  $\Psi_0$  is between these extrema, the streamline with  $\Psi_0$  flows into the fracture. Although the influx point  $z_{in}$  can be found in the  $z$ -plane, the outflux point  $z_{out}$  must be identified in the transformed  $\chi$ -plane [8]. This is because the  $+$  and  $-$  sides of the fracture can be distinguished only in the  $\chi$ -plane. The outflux point  $\chi_{out}$  in the  $\chi$ -plane can be obtained by finding the roots for  $\Psi(\chi_{out}) - \Psi_0 = 0$ . Then,  $\chi_{out}$  is back-transformed to  $z_{out}$  through eqn (5).

### 3.3 Flow simulation along streamlines

Streamline simulation [13] has been used for modeling fluid flow in porous media. The fundamental idea is to decompose a multi-dimensional flow problem into multiple 1D flow problems to be solved along streamlines.

Along the streamline coordinate, the integrands in eqns (8) and (9) can be written as  $\phi dr/|u(r)| = \phi dt$  and  $\phi^2 dr/u^2(r) = \phi^2 dt^2/dr$ , respectively. With an approximation of eqn (17),  $\tau$  and  $\omega$  are evaluated as

$$\begin{cases} \tau(s) = \sum_j \phi \Delta t_j, \\ \omega(\bar{s}) = \sum_j \frac{\phi^2 \Delta t_j^2}{\Delta r_j}. \end{cases} \quad (18)$$

With  $\tau$  and  $\omega$ , the 1D solution for convective-dispersive transport is given by eqn (11). The complete 2D solution can be recovered by combining the individual 1D solutions.



## 4 Results and discussion

### 4.1 Flow through a fracture

Let us consider a square ( $1 \times 1$ ) flow domain with an impermeable boundary, which is represented by arranging 40 equally spaced boundary elements. A fracture of length 0.5 is located at (0.25, 0.5) with an inclination of  $\pi/2$ . A source and a sink are located at (0, 0) and (1, 1), respectively, and injection and production rates are constant and equal to each other.

First,  $k_f w_f / k = 0.0833$  is assumed and the number of fracture parameters is set as  $n_L = n_p = 5$ . Fig. 3 (a) shows 101 streamlines tracked from the source to the sink with  $\Delta s = 0.005$ . The streamlines passing through the fracture are indicated by the black lines. 39 streamlines flow into and out of the fracture.

Next, a fracture with a higher conductivity  $k_f w_f / k = 83.3$  is considered. Fig. 3 (b) shows 101 streamlines. Compared with the previous case, the fracture influence on flow profiles becomes more conspicuous; 55 streamlines flow into and out of the fracture. It should be noted that the lower part of the fracture has influx on both sides, while the upper part has outflux on both sides.

### 4.2 Tracer flow with dispersion

A means to follow fluid movement in porous media is an invaluable tool for evaluating hydrodynamic characteristics. Interwell tracer tests [14] have been conducted for this purpose, in which a radioactive or chemical tracer is injected in a source point and detected at a sink point.

#### 4.2.1 A single fracture

For the same fracture system above ( $k_f w_f / k = 83.3$ ), a tracer slug is injected and is followed in turn by a tracer-free fluid. The concentration profile of the tracer-free fluid is delayed by the dimensionless slug size  $\Delta V_p$ , defined as the ratio of slug volume to pore volume. The tracer-slug concentration profile can be obtained by subtracting the concentration profile of the tracer-free fluid from that of the tracer.

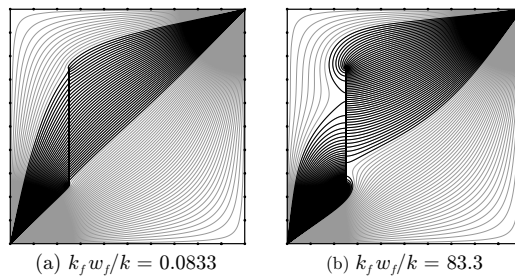


Figure 3: Streamlines through a single fracture.



Fig. 4 shows the tracer-slug concentration profiles at  $V_p = 0.5$  with various dispersivities, where  $V_p$  is the pore volumes injected, defined as the ratio of cumulative injection to pore volume. A tracer slug of  $C_0 = 1$  and  $\Delta V_p = 0.1$  is assumed. As the value of  $\alpha$  increases, the leading and trailing interfaces between the tracer slug and the tracer-free fluid become indistinct.

The effluent tracer concentration  $C(t)$  at the sink can be constructed as

$$C(t) = \frac{\sum_{j=1}^{n_s} C(\tau_{bj}, t) q_j}{\sum_{j=1}^{n_s} q_j}, \tag{19}$$

where  $n_s$  is the number of streamlines,  $C(\tau_{bj}, t)$  is given by eqn (11),  $\tau_{bj}$  is the TOF of the  $j$ -th streamline corresponding to the breakthrough time, and  $q_j$  is the flow rate assigned to the  $j$ -th streamline. Effluent concentration curves for various dispersivities are shown in fig. 5. Due to the single fracture, two peaks are observed: one around  $V_p = 0.6$  and another around  $V_p = 1.0$ . The first peak corresponds to the streamlines passing through the fracture and the second corresponds to those not passing through it. The tracer breakthrough time becomes earlier and the peak concentration becomes lower for larger values of  $\alpha$ .

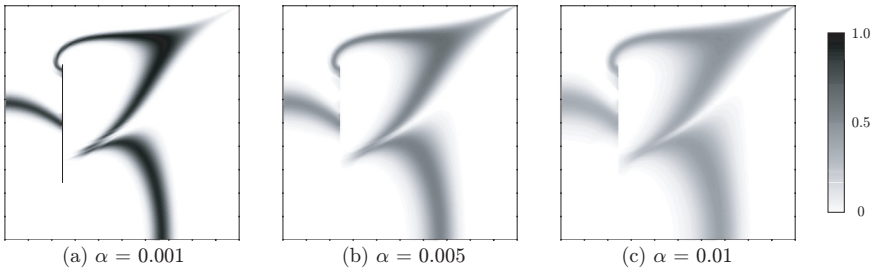


Figure 4: Tracer-slug concentration with various dispersivities.

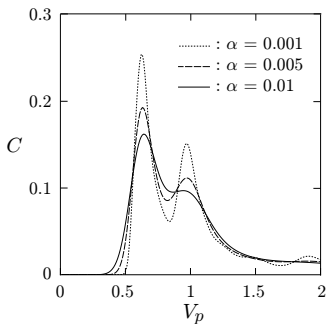


Figure 5: Effluent tracer concentrations for various dispersivities.

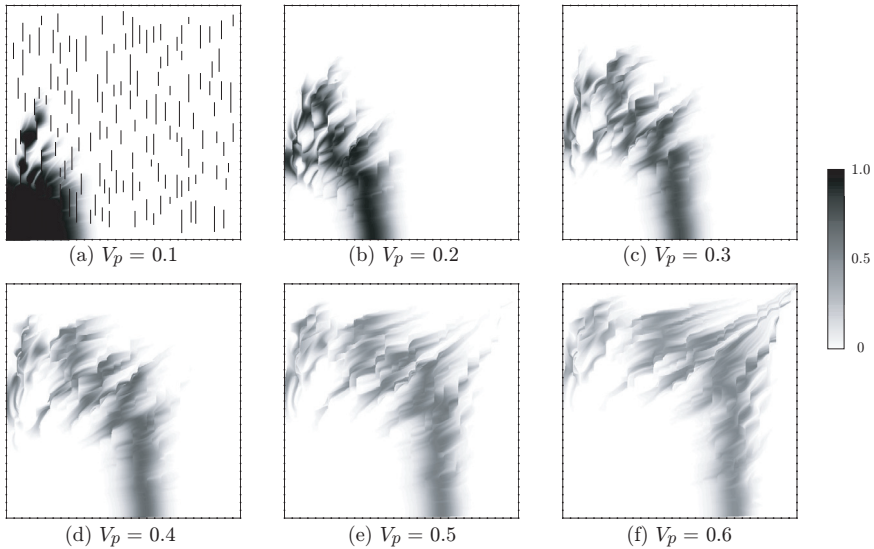


Figure 6: Dispersive tracer-slug flow through distributed fractures.

#### 4.2.2 Distributed fractures

The developed model can simulate multiple fractures. Randomly distributed fractures are considered in the same flow domain, the boundary of which is represented by 120 elements. A stochastic fracture system generated by an unconditional Boolean technique is superimposed on fig. 6 (a), where the number of fractures is 142 and the average fracture length is 0.0764.

Fig. 6 shows the tracer-slug concentration profiles at  $V_p = 0.1$  through 0.6, where  $\alpha = 0.005$ ,  $C_0 = 1$ , and  $\Delta V_p = 0.1$  are assumed. As  $V_p$  increases and the tracer advances, the tracer concentration becomes more dissipated.

## 5 Conclusions

1. A singular solution for fractures is obtained in the conformally mapped plane, which is coupled with the CVBEM. The stream function, evaluated directly with the CVBEM, is used to find the points of a constant value of the stream function. This allows the tracking of the streamlines passing through fractures.
2. Complex 2D flow is decomposed into flow along multiple 1D streamlines. The CVBEM provides accurate particle locations and stepwise time increments, which are used to compute a set of coordinate transformations,  $\tau$  and  $\omega$ . Along each streamline, dispersive flow is modeled with the Gelhar-Collins solution, and the complete solution is recovered by combining the individual 1D solutions.



3. The applicability of the CVBEM has thus been extended to dispersive flow in the presence of distributed fractures. The developed model has been applied successfully to example flow problems. Because the flow domain is not discretized and analytical solutions are used along the streamlines, the results are devoid of numerical dispersion and reasonably smooth.

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