Hyper-singular boundary element method for elastic fracture mechanics analysis with large deformation

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Abstract

In this paper a hyper-singular boundary element method (HBEM) for elastic fracture mechanics analysis with large deformation is presented. The non-linear formulation incorporates the displacement and the traction boundary integral equations as well as finite deformation stress measures. Efficient regularization techniques are applied to the corresponding singular terms in displacement, displacement derivatives and traction boundary integral equations, according to the degree of singularity of the kernel functions. Fracture measures are evaluated at each load increment. Several case studies with different boundary and loading conditions have been analysed. It has been shown that the new singularity removal technique and the non-linear formulation lead to accurate solutions.

Keywords: hypersingular boundary element method; non-linear elastic analysis; fracture mechanics, large deformation, J-integral.

1 Introduction

In crack problems, the boundary element method (BEM) has shown to be a strong numerical technique to provide good engineering accuracy. Some developments in the use of BEM for fracture analysis are given by Cruse [1] and Gray et al [2]. Boundary integral equations (BIEs) with hyper-singular kernels arise whenever the normal derivative of a classical boundary integral equation is taken. Fracture mechanics has been a potential area of research and application for hyper-singular boundary integral equations (HBIEs). Portela et al [3] presented a numerical implementation of the two-dimensional dual boundary element method formulations (DBEM), for solving general linear elastic fracture
mechanics problems. A review of the DBEM for modelling of crack problems in fracture mechanics was presented by Aliabadi [4].

In this paper, the hyper-singular boundary element method (HBEM) is applied to elastic fracture mechanics analysis with finite deformation. Fracture mechanics measures are evaluated at each load increment, using an energy principle, and Rice’s J-integral [5]. Efficient regularization techniques are applied to the corresponding singular terms in different BIEs. Several cases studies with different crack sizes have been analysed.

2 Strain, stress and traction relations for non-linear boundary integral equations

The total strain vector can be partitioned such that the small and large deflection parts can be defined in terms of the displacement components by the Green strain matrix. For boundary element analysis of non-linear problems, the stress vector $\boldsymbol{\sigma}$ is also partitioned, where the stress matrix becomes:

$$
[\boldsymbol{\sigma}] = [\boldsymbol{\sigma}^l] + [\boldsymbol{\sigma}^{nl}]
$$

(1)

where $[\boldsymbol{\sigma}^l]$ is a symmetric matrix. For 2D problems it can be represented by:

$$
\boldsymbol{\sigma}^l = \mathbf{D} \varepsilon^s
$$

(2)

where $\mathbf{D}$ is the stress-strain matrix, and $\varepsilon^s$ is the infinitesimal strain vector. Then, eqn (2) is assumed as a definition for $\boldsymbol{\sigma}^l$ whether the problem is linearly elastic or not. For two-dimensional problems of metals and alloys the $\boldsymbol{\sigma}^{nl}$ matrix can be assumed symmetric and for those cases, and it can be represented by the following vector:

$$
[\boldsymbol{\sigma}^{nl}] = \left\{ \sigma^{nl}_x, \sigma^{nl}_y, \tau^{nl}_{xy} \right\}
$$

(3)

For large deflection linear elasticity:

$$
\boldsymbol{\sigma} = \mathbf{D} \varepsilon = \mathbf{D} \varepsilon^s + \mathbf{D} \varepsilon^L
$$

(4)

i.e.

$$
\boldsymbol{\sigma}^{nl} = \mathbf{D} \varepsilon^L
$$

(5)

For two-dimensional boundary element analysis with finite strains, traction components are related to stress components by means of the following equilibrium equations on the boundary:
where \( l, m \) are direction cosines of the outward normal to the boundary. It is important to notice that \( \tau_{xy}^{l} = \tau_{yx}^{l} \).

### 3 Boundary integral equations

For two-dimensional non-linear elastic problems, the basic equations for the BIE derivation are the equations of equilibrium for linear or non-linear analysis, which are defined with respect to the original dimensions of the structure (i.e. using undeformed or Lagrangian coordinates) as follows:

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y &= 0
\end{align*}
\]  

(7)

where \( f_x \) and \( f_y \) are the domain loading components and \( \sigma_x, \sigma_y, \tau_{xy}, \tau_{yx} \) are the components of the first Piola-Kirchhoff stress measure \( P \). Hence, equations of equilibrium given by eqn (7) can be written in a more general expression as follows:

\[
\frac{\partial P_{ij}}{\partial x_j} + f_i = 0
\]  

(8)

where \( x_j \) are the reference coordinates. The relation between the reference coordinates and the current coordinates \( X_i \), is defined by the deformation gradient matrix, \( G_{ij} = \partial X_i / \partial x_j \). It can be shown that the first Piola-Kirchhoff stress can be expressed in terms of the second Piola-Kirchhoff stress \( S \) so as to deal with elastic problems with finite deformation relating both current and reference coordinates as presented by Jirasek and Bazant [6]:

\[
P = G \cdot S
\]  

(9)

The displacement boundary integral equation with respect to the source point \((x_o, y_o)\) can be obtained by substituting the fundamental solution parameters into the Somigliana’s identity [7], leading to:
\[ C_0 u_\alpha^o = \oint_{\Gamma} \left( \sum_\beta F_{\beta \alpha} u_\beta \right) d\Gamma + \oint_{\Gamma} \left( \sum_\beta G_{\beta \alpha} T_\beta \right) d\Gamma \]
\[ + \iint_{\Omega} \left( \sum_\beta G_{\beta \alpha} f_\beta \right) dx dy - \sum_\beta \sum_\delta \iint_{\Omega} \frac{\partial G_{\beta \alpha}}{\partial x_\delta} \sigma_{\delta \beta}^{nl} dx dy \]

where \( u_\alpha^o \equiv u_\alpha(x_o, y_o) \); \( \alpha = 1, 2 \); \( \beta = 1, 2 \); \( (x_1, x_2) \equiv (x, y) \);
\((u_1, u_2) \equiv (u, v)\); \((f_1, f_2) \equiv (f_x, f_y)\) and \((T_1, T_2) \equiv (T_x, T_y)\).

In eqn (10), the parameters \( u_\beta, T_\beta, f_\beta, \sigma_{\delta \beta}^{nl} \) are functions of the field point \((x, y)\) within the integration region \( \Gamma \) or \( \Omega \), whilst all kernel functions such as \( F_{\beta \alpha}, G_{\beta \alpha} \) are functions of \( r, \partial r/\partial x, \partial r/\partial y \) i.e. functions of \((x - x_o), (y - y_o)\), where \( r \) is the distance between field and source points. If a source point \((x_o, y_o)\) is on a smooth boundary, it can be proven that the boundary integral equations of displacement derivatives are given by:

\[ \frac{1}{2} \frac{\partial u_\beta^o}{\partial x_\gamma} = \oint_{\Gamma} \left( \sum_\beta \frac{\partial F_{\beta \alpha}}{\partial x_\gamma} (u_\beta - u_\beta^o) \right) d\Gamma - \oint_{\Gamma} \left( \sum_\beta \frac{\partial G_{\beta \alpha}}{\partial x_\gamma} T_\beta \right) d\Gamma \]
\[ - \iint_{\Omega} \left( \sum_\beta \frac{\partial G_{\beta \alpha}}{\partial x_\gamma} f_\beta \right) dx dy + \sum_\beta \sum_\delta \iint_{\Omega} \frac{\partial^2 G_{\beta \alpha}}{\partial x_\gamma \partial x_\delta} \sigma_{\delta \beta}^{nl} dx dy \]

It should be noticed that the first boundary integral term and the last domain integral of the RHS in eqn (11) have \((1/r^2)\) singularity. On the other hand, the other integral terms have \((1/r)\) singularity.

For boundary element analysis of non-linear problems, the stress vector \( \sigma \) is usually partitioned as given by eqn (3), where the stress matrix that includes the non-linear behaviour can be given by eqn (5). The stress-strain relationship with the large deflection part of the total strain is given by eqns (4) and (5).

Traction components \((T_x^o, T_y^o)\) evaluated at a source point, which is on a smooth part of the boundary \( \Gamma \) of a two-dimensional domain \( \Omega \) can be obtained by substituting from eqn (2) into the equilibrium eqns given by eqn (6) and applying the resulting equation at a source point \((x_o, y_o)\). Substituting from the explicit form of eqn (11) into the traction component equations of \((T_x^o, T_y^o)\) it can be proven that a general boundary integral equation for tractions at a source.
point \( (x_o, y_o) \), which is on a smooth part of the boundary \( \Gamma \) of a two-dimensional domain \( \Omega \), can be expressed as follows:

\[
\frac{1}{2} \mathbf{T}_a^o = \oint_\Gamma \left( \sum_{\beta} A_{\beta \alpha} \left( u_\beta - u_\beta^o \right) \right) d\Gamma - \oint_\Gamma \left( \sum_{\beta} B_{\beta \alpha} T_\beta \right) d\Gamma
\]

\[
- \iint_\Omega \left( \sum_{\beta} B_{\beta \alpha} f_\beta \right) dx dy + \iint_\Omega \left[ \sum_{\beta} \sum_{\gamma} \frac{\partial B_{\beta \alpha}}{\partial x_\gamma} n^l_\beta \sigma_{\gamma \beta} \right] dx dy + \frac{1}{2} \left( \mathbf{T}_a^nl \right)_o
\]

where

\[
A_{\alpha \beta} = \frac{\mu}{2\pi(1 - p) r^2} \left\{ 8 \frac{\partial r}{\partial n_o} \frac{\partial r}{\partial x_\alpha} - \frac{\partial r}{\partial x_\beta} \delta_{\alpha \beta} \left[ (\hat{n} \cdot \hat{n}_o) - 2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial n_o} \right] - \left( l_o l_\beta^o + l_\beta l_o^o \right) - 2(\hat{n} \cdot \hat{n}_o) \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right\}
\]

\[
B_{\alpha \beta} = -\frac{1}{4\pi(1 - p) r} \left\{ 2 \frac{\partial r}{\partial n_o} \frac{\partial r}{\partial x_\alpha} + (1 - 2 p) \left[ \frac{\partial r}{\partial n_o} \delta_{\alpha \beta} \right] + l_o \frac{\partial r}{\partial x_\beta} - l_\beta^o \frac{\partial r}{\partial x_\alpha} \right\}
\]

and \( \mu \) is shear modulus, \( p \) is a modified Poisson’s ratio [7],

\[
\frac{\partial r}{\partial n_o} = l_o \frac{\partial r}{\partial x} + m_o \frac{\partial r}{\partial y}
\]

\[
(\hat{n} \cdot \hat{n}_o) = l_o + m o
\]

\[
\frac{\partial B_{\alpha \beta}}{\partial x_\gamma} = \frac{1}{4\pi(1 - p) r^2} \left\{ 8 \frac{\partial r}{\partial n_o} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial x_\gamma} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \left( \hat{i}_\gamma \cdot \hat{n}_o \right)
\]

\[
- 2 \frac{\partial r}{\partial x_\beta} \frac{\partial r}{\partial n_o} \delta_{\alpha \gamma} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial n_o} \delta_{\beta \gamma}
\]

\[
- (1 - 2 p) \left[ (\hat{i}_\gamma \cdot \hat{n}_o) \delta_{\alpha \beta} + l_o \delta_{\beta \gamma} - l_\beta \delta_{\alpha \gamma} \right]
\]

\[
+ 2(1 - 2 p) \left[ \frac{\partial r}{\partial n_o} \delta_{\alpha \beta} + l_o \frac{\partial r}{\partial x_\beta} - l_\beta \frac{\partial r}{\partial x_\alpha} \right]
\]
It should be noticed that generalized expressions for $A_{\alpha\beta}$ and $\partial B_{\alpha\beta}/\partial x_\gamma$ include $(1/r^2)$ singularity, and that for $B_{\alpha\beta}$ includes $(1/r)$ singularity.

4 Treatment of integrals using regularization techniques and Cauchy principal value theorem

Evaluation of singular integrals is one of the major problems while implementing non-linear boundary element techniques. Singularity removal techniques and Cauchy principal value theorem were used to deal with singularities of the form $(1/r)$ and $(1/r^2)$ appearing in displacement, displacement derivatives and traction boundary integral equations. Considering the boundary integral equation for tractions, the fourth term in eqn (12) has hyper-singularity $(1/r^2)$. The starting point of the regularization is carried out by defining

$$D^o_{\alpha} = \int\int_{\Omega} \sum_{\beta} \sum_{\gamma} \frac{\partial B_{\beta\alpha}}{\partial x_\gamma} \sigma^{nl}_{\gamma\beta} \, dx \, dy \tag{18}$$

where $(D^1_{\alpha}, D^2_{\alpha}) \equiv (D^o_{x}, D^o_{y})$, represent the domain integral terms due to non-linear stresses. Then eqn (18) can be modified as follows:

$$D^o_{\alpha} = \sum_{\beta} \sum_{\gamma} \int\int_{\Omega} \frac{\partial B_{\beta\alpha}}{\partial x_\gamma} \left[ \sigma^{nl}_{\gamma\beta} - \left( \sigma^{nl}_{\gamma\beta} \right)_o \right] \, dx \, dy \tag{19}$$

where $\left( \sigma^{nl}_{\gamma\beta} \right)_o$ is the value of $\left( \sigma^{nl}_{\gamma\beta} \right)$ at the source point $(x_o, y_o)$, hence the second term on the right-hand side of eqn (19) can be written as:

$$\sum_{\beta} \sum_{\gamma} \int\int_{\Omega} \frac{\partial B_{\beta\alpha}}{\partial x_\gamma} \left( \sigma^{nl}_{\gamma\beta} \right)_o \, dx \, dy = \sum_{\beta} \sum_{\gamma} \left[ \left( \sigma^{nl}_{\gamma\beta} \right)_o \int\int_{\Omega} \frac{\partial B_{\beta\alpha}}{\partial x_\gamma} \, dx \, dy \right] \tag{20}$$

Defining:

$$\tilde{T}^{nl}_{\beta} = \sum_{\gamma} l_{\alpha} \left( \sigma^{nl}_{\gamma\beta} \right)_o \tag{21}$$
\[ \tilde{T}^{nl}_x = l \left( \sigma^{nl}_x \right)_o + m \left( \tau^{nl}_{yx} \right)_o \]
\[ \tilde{T}^{nl}_y = l \left( \tau^{nl}_{xy} \right)_o + m \left( \sigma^{nl}_y \right)_o \] (22)

and defining the non-linear traction components at \((x_o, y_o)\), as follows:

\[
\begin{align*}
\left( T^{nl}_x \right)_o &= l_o \left( \sigma^{nl}_x \right)_o + m_o \left( \tau^{nl}_{yx} \right)_o \\
\left( T^{nl}_y \right)_o &= l_o \left( \tau^{nl}_{xy} \right)_o + m_o \left( \sigma^{nl}_y \right)_o
\end{align*}
\] (23)

then eqn (19) can be rewritten as follows:

\[ D^o_a = \sum_{\beta} \int \int_{\Omega} \int \int_{\Gamma} \frac{\partial G_{\beta\alpha}}{\partial x_{\gamma}} \left[ \sigma^{nl}_{\gamma\beta} - \left( \sigma^{nl}_{\gamma\beta} \right)_o \right] dx \, dy + \sum_{\beta} \oint_{\Gamma} B_{\beta\alpha} \left( \tilde{T}^{nl}_\beta \right) d\Gamma \] (24)

Hence the traction BIE, can be modified by substituting from eqn (24) into the eqn (12) as follows:

\[ \frac{1}{2} T^o_\alpha = \oint_{\Gamma} \left[ \sum_{\beta} A_{\beta\alpha} \left( u_\beta - u^o_\beta \right) \right] d\Gamma \\
- \oint_{\Gamma} \left[ \sum_{\beta} B_{\beta\alpha} \left( T_\beta - \tilde{T}^{nl}_\beta \right) \right] d\Gamma - \int \int_{\Omega} \left[ \sum_{\beta} B_{\beta\alpha} f_\beta \right] dx \, dy \\
+ \int \int_{\Omega} \left[ \sum_{\beta} \sum_{\gamma} \frac{\partial B_{\beta\alpha}}{\partial x_{\gamma}} \left[ \sigma^{nl}_{\gamma\beta} - \left( \sigma^{nl}_{\gamma\beta} \right)_o \right] \right] dx \, dy + \frac{1}{2} \left( T^o_\alpha \right)_o \] (25)

5 Numerical implementation of the hyper-singular boundary element method (HBE M) for fracture mechanics problems

The numerical implementation of the hyper-singular boundary element method for fracture mechanics problems presented in this paper is based upon an efficient regularization method for hyper-singular integrals, an effective handling of residual or non-linear stress components and finally on a formulation for the boundary element method (BEM) able to deal with J-integral calculations.

Two important aspects are considered in the numerical implementation: (a) the stress correction not only at boundary nodes but also at internal points within internal cells inside the domain and (b) the force correction at any given internal point. The energy balance approach was implemented to evaluate the fracture
parameter $J$ for a given crack extension by computing energy changes due to crack growth, and was compared with $J$ computations from contour integral approach.

6 Numerical examples

The accuracy of the proposed formulation to deal with the regularization of singular boundary and domain integrals was evaluated by considering numerical examples that include different types of loading. Two plane stress fracture mechanics problems such as centre-cracked plate, and double-edge cracked plate were evaluated. The material properties were Young’s modulus $E = 8.0 \times 10^4$ N/mm$^2$, Poisson’s ratio $\nu = 0.3$. The applied load was $\sigma_o = 100$ N/mm$^2$. Each model was analysed using the hypersingular boundary element analysis technique using three-node quadratic elements over the boundary and eight-node quadrilateral isoparametric elements were used for the internal cell mesh. Fracture mechanics measures versus crack size are shown in figures 1 and 2. In addition, in order to evaluate the approach here presented with domain type loading, a centre cracking rotating disc (radius=100 mm) case was analyzed. The material properties were Young’s modulus $E = 2.25 \times 10^5$ N/mm$^2$, Poisson’s ratio $\nu = 0.25$. Results are given in figure 3 at 81rad/s. In this case, the sectorial symmetry concept that reduces the problem associated with large number of boundary and internal cells in boundary element models was introduced. Good agreement with the analytical results was obtained for all cases.

![Figure 1: J-fracture mechanics parameter versus crack length for a centre cracked plate.](image-url)
Figure 2: J-fracture mechanics parameter versus crack length for a double-edge cracked plate.

Figure 3: J-fracture mechanics parameter versus crack length for a centre cracked rotating disc.
7 Conclusions

The success of the formulation presented in this work was the application of a simple approach to the regularization of the domain integrals and the application of the Cauchy principal value theorem to reduce domain and boundary integrals from displacement derivatives and traction boundary integral equations. In this work, boundary integral equations were used to evaluate the stresses at given internal points except at corners. This work also introduces a hypersingular boundary element method of analysis which applies the sectorial symmetry concept that reduces the problem associated with large number of boundary and internal cells in boundary element models. Implementation of the energy balance approach to evaluate the fracture parameter $J$ for a given crack extension by computing energy changes due to crack growth shown to be accurate when compared with $J$ computations from J-Integral approach. This work has been developed in order to advance towards the improvement of the efficiency, accuracy and robustness of the boundary element and hypersingular boundary element approaches for non-linear elastic problems involving finite deformation.

Acknowledgements

J.M. Franco-Nava wishes to acknowledge the financial support provided by the Mexican Council of Science and Technology, (Consejo Nacional de Ciencia y Tecnologia, CONACyT, México) for the realization of this work. In addition, J.M. Franco-Nava acknowledges the complementary financial support provided through the Bank of Mexico for the Electric Research Institute (Instituto de Investigaciones Eléctricas, IIE, México).

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