

The method of fundamental solutions, a dipole formulation for potential problems

G. S. A. Fam¹ & Y. F. Rashed²

¹*Wessex Institute of Technology, Southampton, UK*

²*Dept. of Structural Engineering, Cairo University, Giza, Egypt*

Abstract

This paper introduces the use of a dipole formulation within the Method of Fundamental Solutions for potential problems. The present formulation is set up by taking the limiting case of two adjacent sources. The necessary kernels are derived in explicit forms. Two numerical examples, including torsion analysis of irregular cross section, are solved. Several parametric studies are presented to demonstrate different configurations for the placement of sources (monopoles or dipoles). The accuracy of the present new formulation is verified by comparing its results to those obtained from the traditional source formulation.

1 Introduction

The Method of Fundamental Solutions (MFS) is an indirect discrete boundary integral method. It appeared 4 decades ago in the work of Kupradze and Alkesidze [1] as The Method of Functional Equations. It was then introduced to various problems in different fields including Potential [2],[3],[4], Acoustics [5], Fluid dynamics [6], and Elasticity [7],[8].

This method is distinguished by its mathematical simplicity, in addition to being a boundary method (no need to perform a complete domain discretization). The major drawback, which kept research limited in this method, is the ill-conditioned nature of the system of equations obtained through collocation. Nowadays, by the aid powerful computers and advanced programming tools, this problem started to be overcome, and research in MFS has remarkably increased during the last few years.

In the following sections, the formulation for potential problem with monopoles is reviewed briefly, followed by a detailed derivation of the new



formulation including dipole sources. Some numerical examples covering irregular geometries and boundary conditions are solved using the presented formulation with different techniques for placement and types of sources. The obtained results are discussed and finally, the conclusions are drawn.

2 The MFS formulation

In the MFS formulation, the problem domain can be considered as imbedded in an infinite domain where the solution domain is defined by the conditions prescribed along its boundaries (the boundary conditions). The variables of the problem are defined by means of fictitious sources. These sources can be placed theoretically anywhere, and their intensities are tuned through collocation at different boundary points to represent the known boundary conditions. Once set, these fictitious sources can be used to determine the values of any unknowns in the solution domain using considerably simple mathematical operations.

It has to be noted that the effect of a source of a unit intensity placed at any point ξ_i to any field point X_j is the fundamental solution kernel between ξ_i and X_j ($u^*(X_j, \xi_i)$)

2.1 The MFS formulation using monopoles

The potential u at point X in a homogeneous isotropic medium satisfies Laplace equation:

$$\nabla^2 u(X) = 0 \quad (1)$$

The potential at any point X inside the domain Ω can be represented as the sum of the potential due to all sources:

$$u(X_j) = \sum_{i=1}^n \left(u^*(X_j, \xi_i) \times \varphi(\xi_i) \right) \quad (2)$$

where,

$u(X_j)$ is the potential at point X_j

$u^*(X_j, \xi_i)$ is the potential at point X_j due to the effect of a unit source at ξ_i

$$u^*(X_j, \xi_i) = -\frac{1}{2\pi} \ln r(X_j, \xi_i) \quad (3)$$

$r(X_j, \xi_i)$ is the Euclidean distance between point X_j and source ξ_i

$\varphi(\xi_i)$ is the potential density.

Also the flux q (potential gradient) distribution can be represented as the normal derivative of the potential

$$q(X_j) = \sum_{i=1}^n \left(q^*(X_j, \xi_i) \cdot \varphi(\xi_i) \right) \quad (4)$$

where,

$q(X_j)$ is the flux at point X_j



$$q^*(X_j, \xi_i) = -\frac{1}{2\pi r(X_j, \xi_i)} \cdot [r_{,1} n_1 + r_{,2} n_2] \quad (5)$$

$$r_{,k} = \frac{\partial r}{\partial x_k} = \frac{r_k}{r} \quad (6)$$

r_k is the component of r in the direction of x_k

$r_{,k}$ is a vectorial quantity, and the direction is always from the source to ξ_i the point X_j

n_k is the component of the normal outward vector n at point X_j in the direction of x_k

Writing equations (2) and (4) in their expanded form for different boundary points X_j with known potential or flux, the following system of equations is obtained:

$$\begin{pmatrix} uq^*(X_1, \xi_1) & uq^*(X_1, \xi_2) & \cdots & uq^*(X_1, \xi_n) \\ uq^*(X_2, \xi_1) & & & \\ \vdots & & & \\ uq^*(X_m, \xi_1) & \cdots & \cdots & uq^*(X_m, \xi_n) \end{pmatrix} \begin{pmatrix} \varphi(\xi_1) \\ \varphi(\xi_2) \\ \vdots \\ \varphi(\xi_n) \end{pmatrix} = \begin{pmatrix} uq(X_1) \\ uq(X_2) \\ \vdots \\ uq(X_m) \end{pmatrix} \quad (7)$$

where,

$uq^*(X_j, \xi_i)$ is the potential u^* , or the flux q^* , at point X_j due to the effect of a unit source at ξ_i

$uq(X_j)$ is the total potential u , or the flux q , at point X_j

Hence, by solving the former system of equation (7), the sources intensities $\varphi(\xi_i)$ are set, for any $i = 1$ to n , and equations (2) and (4) can be used to calculate the value of the potential (u) or the flux (q) at any point X_j in the solution domain Ω .

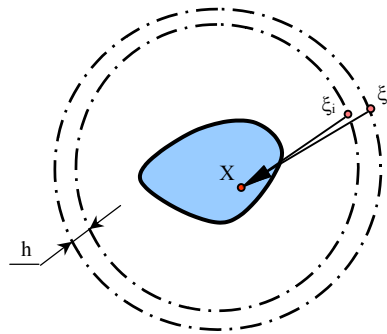


Figure 1: Dipoles derivation.

2.2 The MFS formulation using dipoles

Using the formulation in section 2.1, and placing the sources on 2 surfaces (with arbitrary shape) distant by a constant distance h as shown in figure 1

$$u(X_j) = \sum_{i=1}^n \left(u^*(X_j, \xi_i) \cdot \varphi(\xi_i) \right) + \sum_{i=1}^n \left(u^*(X_j, \xi'_i) \cdot \varphi(\xi'_i) \right) \quad (8)$$

where ξ_i are the sources placed on one surface, while ξ'_i are the sources placed on the other surface.

Taking $\varphi(\xi_i) = -\varphi(\xi'_i)$

$$\begin{aligned} u(X_j) &= \sum_{i=1}^n \left(u^*(X_j, \xi_i) - u^*(X_j, \xi'_i) \right) \varphi(\xi_i) \\ &= \sum_{i=1}^n \left(\frac{u^*(X_j, \xi_i) - u^*(X_j, \xi'_i)}{h} \right) h \varphi(\xi_i) \end{aligned} \quad (9)$$

by replacing $h \varphi(\xi_i)$ by $\psi(\xi_i)$ and taking the limit when h tends to zero,

$$\begin{aligned} u_d(X_j) &= \sum_{i=1}^n \lim_{h \rightarrow 0} \left(\frac{u^*(X_j, \xi_i) - u^*(X_j, \xi'_i)}{h} \right) \psi(\xi_i) \\ &= \sum_{i=1}^n \left(\frac{\partial}{\partial n_\alpha(\xi_i)} u^*(X_j, \xi_i) \right) \psi(\xi_i) \end{aligned} \quad (10)$$

$$u_d(X_j) = \sum_{i=1}^n \left(\frac{\partial}{\partial x_\alpha(\xi_i)} u^*(X_j, \xi_i) \right) n_\alpha(\xi_i) \cdot \psi(\xi_i) \quad (11)$$

substituting with $n_\alpha(\xi_i) \cdot \psi(\xi_i) = \psi_\alpha(\xi_i)$ in (11)

$$u_d(X_j) = \sum_{i=1}^n \left(\frac{\partial}{\partial x_\alpha(\xi_i)} u^*(X_j, \xi_i) \right) \psi_\alpha(\xi_i) \quad (12)$$

Substituting from (3)

$$\begin{aligned} u_d(X_j) &= \sum_{i=1}^n \left(\frac{\partial}{\partial x_\alpha(\xi_i)} \left(-\frac{1}{2\pi} \ln r(X_j, \xi_i) \right) \right) \psi_\alpha(\xi_i) \\ u_d(X_j) &= \sum_{i=1}^n \left(\frac{r_{,\alpha}(X_j, \xi_i)}{2\pi r(X_j, \xi_i)} \right) \psi_\alpha(\xi_i) \end{aligned} \quad (13)$$

Similarly, the flux kernel can be obtained as the normal derivative of the potential kernel. Hence the flux q_d at X_j can be calculated as follows

$$q_d(X_j) = \frac{\partial}{\partial n_\beta(X_j)} \left(\sum_{i=1}^n \left(\frac{r_{,\alpha}(X_j, \xi_i)}{2\pi r(X_j, \xi_i)} \right) \psi_\alpha(\xi_i) \right)$$

$$\begin{aligned}
&= \sum_{i=1}^n \left(\frac{\partial}{\partial X_{\beta}} (X_j) \left(\frac{r_{,\alpha}(X_j, \xi_i)}{2\pi r(X_j, \xi_i)} \right) \times \psi_{\alpha}(\xi_i) \right) \times n_{\beta}(X) \\
&= \sum_{i=1}^n \left(\frac{1}{2\pi r^2(X_j, \xi_i)} (\delta_{\alpha\beta} - 2 r_{,\alpha}(X_j, \xi_i) r_{,\beta}(X_j, \xi_i)) \psi_{\alpha}(\xi_i) \right) n_{\beta}(X_j) \\
q_d(X_j) &= \sum_{i=1}^n \left(\frac{1}{2\pi r^2(X_j, \xi_i)} (n_{\alpha} - 2 r_{,\alpha}(X_j, \xi_i) r_{,n}(X_j, \xi_i)) \psi_{\alpha}(\xi_i) \right) \quad (14)
\end{aligned}$$

where, $r_{,n}(X_j, \xi_i) = r_{,\beta}(X_j, \xi_i) n_{\beta}(X_j)$
 $\delta_{\alpha\beta} n_{\beta}(X) = n_{\alpha}(X).$

Using the above mentioned formulation (13) and (14), and collocating at different boundary points, the system of equations has the following form

$$[M]\{\Phi\} = \{uq\} \quad (15)$$

where,

$$\begin{aligned}
[M] &= \begin{bmatrix} uq_{11}^*(X_1, \xi_1) & uq_{12}^*(X_1, \xi_1) & uq_{21}^*(X_1, \xi_1) & uq_{22}^*(X_1, \xi_1) & uq_{11}^*(X_1, \xi_2) & \cdots & uq_{22}^*(X_1, \xi_n) \\ uq_{11}^*(X_2, \xi_1) & uq_{12}^*(X_2, \xi_1) & & & & & \\ & & \ddots & & & & \\ \vdots & & & & & & \vdots \\ uq_{11}^*(X_m, \xi_1) & & & & & \cdots & uq_{22}^*(X_m, \xi_n) \end{bmatrix} \\
\{\Phi\} &= [\varphi_{11}(\xi_1) \quad \varphi_{12}(\xi_1) \quad \varphi_{21}(\xi_1) \quad \varphi_{22}(\xi_1) \quad \varphi_{11}(\xi_2) \quad \cdots \quad \varphi_{22}(\xi_n)]^T \\
\{uq_d\} &= [uq_d(X_1) \quad uq_d(X_2) \quad \cdots \quad uq_d(X_3)]^T
\end{aligned}$$

The obtained intensities for the fictitious multi-poles can be used in combination with equations (13) and (14) to calculate the potential or the flux values at any point of the domain.

3 Numerical examples

In this section, two examples, including different types and distribution of boundary conditions and also geometry, are solved using the formulations presented in section 2.

The sources in the following examples, although they could be placed arbitrary, were placed around the domain as a magnification of the boundary points (figure 2). The ratio between the dimension of the sources fictitious surface and that of the domain boundaries (magnification ratio) was always taken bigger than 1 (to keep the sources outside the domain). The values chosen for this ratio varied between 1.5 and 1000 times. The sources were used in 4 different arrangements (as shown in figure 2):



- 1- monopoles only (M),
- 2- dipoles only (D),
- 3- a combination of monopoles and dipoles is used such that at each point there are one monopoles and one dipole (MDS).
- 4- a combination of monopoles and dipoles is used where the dipoles are placed on a fictitious surface with magnification ratio 1.5 times the ratio for the monopoles surface (MDD).

The number of sources was chosen in all cases to give a determined system of equation (square coefficient matrix).

The results obtained for the following examples covers the accuracy of the potential and flux values, and also the system condition number. A useful parameter λ was used for measuring the oscillation in the intensities of the fictitious sources. This parameter was introduced by Fenner [3], and is calculated as $\lambda = \Sigma |\varphi(\xi)| / \Sigma \varphi(\xi)$.

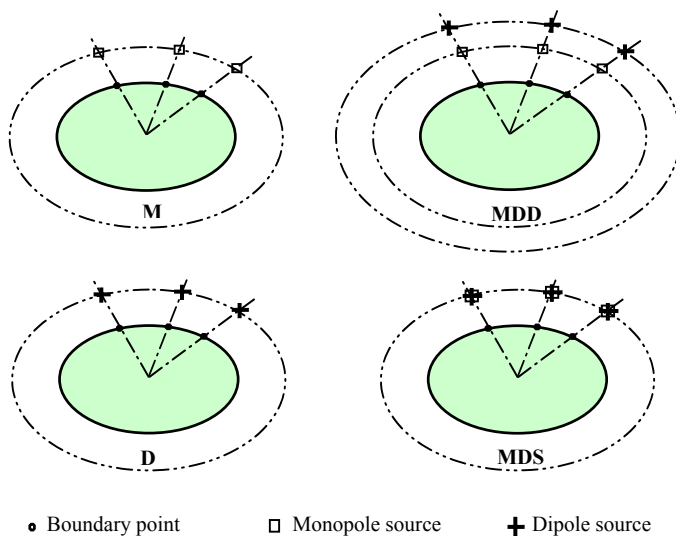


Figure 2: Different sources arrangements.

Example 1.

A rectangular domain of dimensions $1 \times \pi/2$ is considered (figure 3). Dirichlet conditions with sinusoidal distribution of $u = \cos y$ and $u = e \cos y$ are applied on two opposite sides at $x = 0$ and $x = 1$ respectively, while on the other sides a constant potential distribution $u = 0$ at $y = \pi/2$ and a zero flux at $y=0$. The analytical solution (as mentioned by Fenner [3]), at any point inside the domain, is determined by: $u = e^x \cos y$.

Several discretizations were implemented ranging from 4 to 40 boundary points. The magnification ratio for the sources placement was varying from 2 to 1000. Acceptable results were obtained with maximum absolute error of about 0.1. The results presented here (figures 4-5) are obtained in the cases of 12 and 24 boundary points and with magnification ratio was chosen at a value of 5. In

figure 4, the average of the absolute value of the error obtained overall the domain boundaries is shown for both potential and flux. It is clear that for higher discretization (24points) the solution converges towards the analytical solution faster with D 24 and MDD 24 rather than with M 24 or MDS 24.

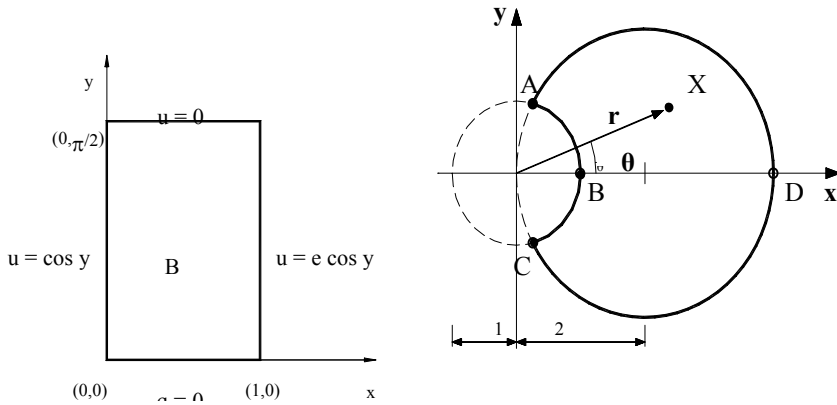


Figure 3: Example 1(left) and Example 2 (right)-Domain and boundary conditions.

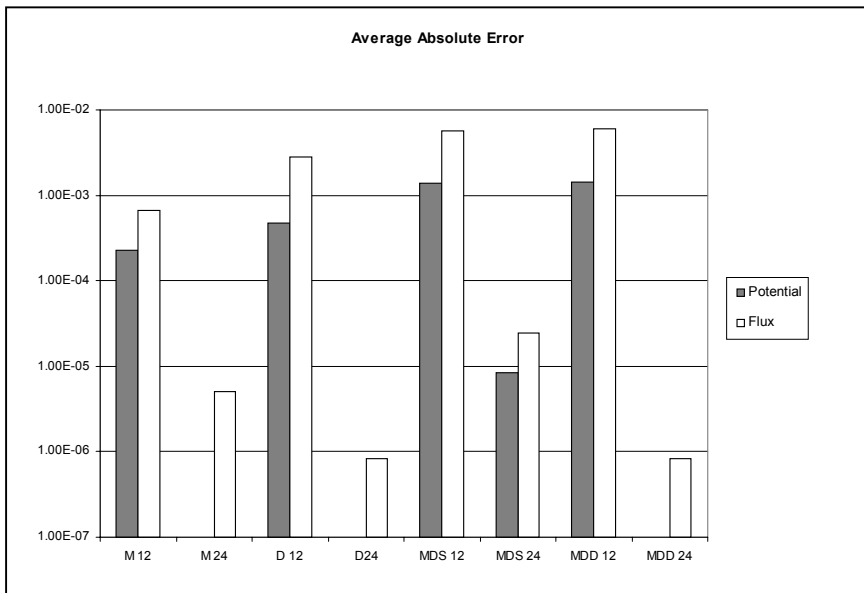


Figure 4: Example 1 - Average absolute error for potential and flux.



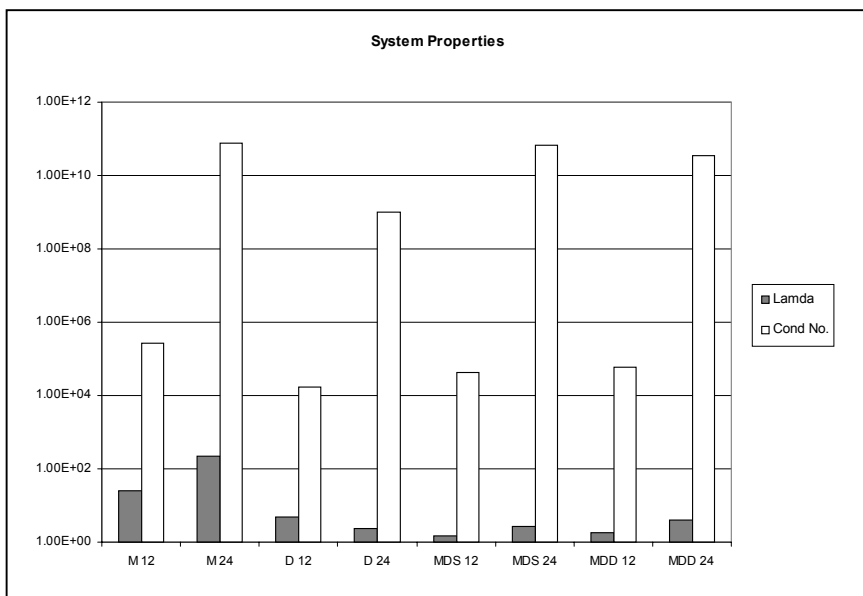


Figure 5: Example 1 - Condition number and oscillation measure.

In figure 5, the condition number is increasing significantly with the discretization, but it is clear that the system of equation formed using dipole sources have lower condition number. The oscillation measure was highly reduced by the use of dipoles alone or in combination with monopoles.

Example 2.

The cross section shown in figure 3 is subjected to torsion. According to Saint-Venant [9], the warping ψ satisfies Laplace equation (1). This example was considered By Paris and Canas [10] and the analytical solution at any point X in the solution domain can be calculated as $\psi(X) = 2(r + 1/r) \sin \theta$ (where, r and θ are shown in figure 3). The boundary conditions are defined in terms of the warping function derivative (\bar{q}) w.r.t. the outward normal (Neumann conditions), where $\bar{q} = 0$ on the arc ABC and $\bar{q} = y$ on the arc CDA. Due to the fully Neumann nature of the boundary conditions, at least 1 point had to be defined with Dirichlet conditions to avoid rigid body motion in the solution. Hence, point B was chosen with zero value for the warping function.

The solution was obtained using different discretizations starting from 41 boundary points (20 with Neumann conditions on each arc + 1 with Dirichlet at point B) and then it was increased gradually up to 202 points (always with 1 Dirichlet conditions and the rest is Neumann). In figures 6-7, the results are shown for the case of 121 and 201 points with monopoles sources (M 121 – M 201), 122 and 202 points with dipoles (D 122 – D 202), and finally 123 and 201 points in case of monopoles + dipoles placed at same points (MDS 123 – MDS 201) or placed on different surfaces (MDD 123 – MDD 201). The magnification ratio of the sources for these results was 2.5.

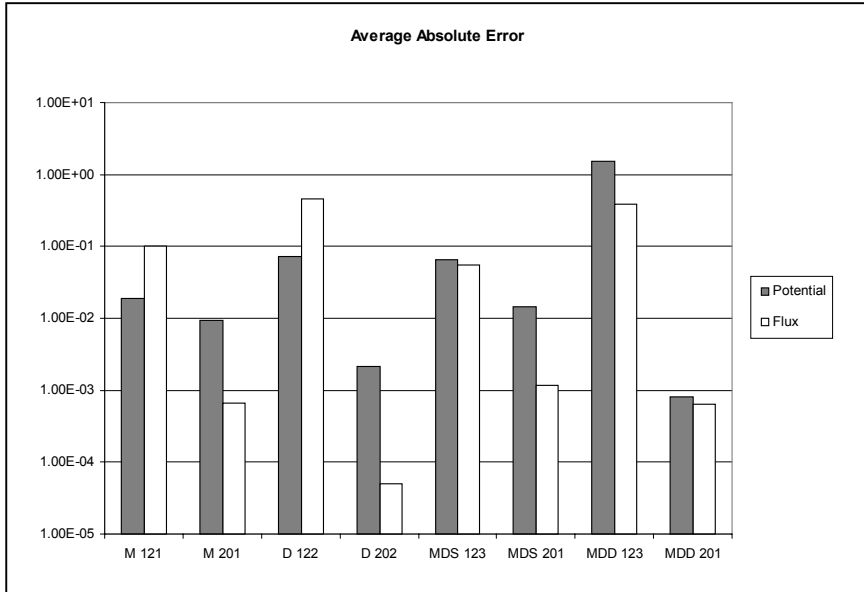


Figure 6: Example 2 - Average absolute error for potential and flux.

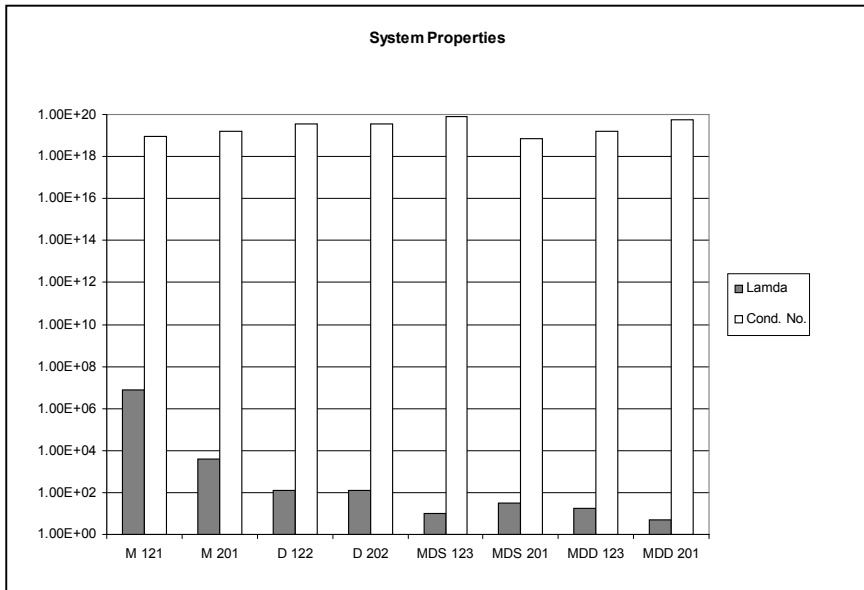


Figure 7: Example 2 - Condition number and oscillation measure.



The plot in figure 6 shows an average of the absolute value of the error obtained at 200 post-processing points overall the domain boundaries for both potential and flux. It is clear that for higher discretization the solution converges towards the analytical solution faster with D 202 and MDD 201 than with M 201 or MDS 201.

In figure 7, the condition number is almost constant for different types of sources. However, the oscillation measure was significantly reduced by the use of dipoles alone and even more reduced when a combination of monopoles with dipoles is used.

4 Conclusions

A new formulation for potential problems using dipoles was introduced within the context of the MFS. The derivation of this formulation was shown thoroughly. A combined approach involving monopoles and dipoles with different relative positioning was introduced. These new formulations and techniques were tested and evaluated by different examples involving various geometries and boundary conditions.

The following conclusions were drawn:

- 1- The derived formulation for dipoles as well as the combined approach are valid and can represent the distribution of the solution accurately.
- 2- In case of the use of dipoles only (D) or combined with monopoles but at different fictitious surfaces (MDD), the numerical solution for potential and flux values, converges with the increase in the discretization much faster than in the other cases (M or MDS).
- 3- The oscillation in the intensities of the sources is largely reduced by the use of dipoles (D) or the combined monopoles and dipoles (MDS or MDD).
- 4- The dipoles doesn't increase the ill conditioning of the coefficient matrix, on the contrary, it reduces it in some cases.

Acknowledgement

This research is supported by the EPSRC grant number GR/R45505.

References

- [1] Kupradze, V. D., Alkesidze, M. A. ,The Method of Functional Equations for the Approximate Solution of Certain Boundary Value Problems, *USSR Computational Mathematics and Mathematical Physics*, 4, pp.82-126, 1964.
- [2] Fairweather, G. , Johnston, R. L., The Method of Fundamental solutions for Problems in Potential Theory, *Proc Symp on Treatment of Integral Equations by Numerical Methods*, 1982.



- [3] Fenner, R. T., Source Field Superposition Analysis of Two Dimensional Potential Problems, *International J. for Numerical Methods in Engineering*, Vol. 32, pp.1079-1091, 1991.
- [4] Saavedra, Ivan, Power, Henry, Adaptive Refinement in Method of Fundamental Solutions for Bidimensional Laplace Problems, *Boundary Element Communications*, Vol 12, No. 1, 2001.
- [5] Koopmann, G. H., A Method for Computing Acoustic Fields Based on the Principle of Wave Superposition, *L. Song, J. B. Fahnlne, J. Acoust. Soc. Am.*, Vol. 86, No. 6, Acoustical Society of America, 1989.
- [6] Strek, Tomasz, Kolodziej, Jan A., Simulation of flows of two fluids in rectangular cavity by the Method of Fundamental Solutions, *Numer. Methods in Continuum Mechanics*, Liptovský Ján, 2000.
- [7] Karageorghis, A., Fairweather, G., *The Method of Fundamental Solutions for Axisymmetric Elasticity Problems*, Computational Mechanics, Vol. 25, pp. 524-532, Springer-Verlag, 2000.
- [8] Fenner, R. T., Force field Superposition Analysis of Three-Dimensional Stress Concentration, *Journal of Strain Analysis*, Vol. 37, No. 5, 2002.
- [9] Timoshenko, S. P., Goodier, J. N., *Theory of Elasticity*, 3rd Ed., McGraw-Hill, pp. 291-303, 1970.
- [10] Paris, F., Cañas, J., *Boundary Element Method, Fundamental and Applications*, Oxford University Press, pp. 162-168, 1997.

