

Modelling and simulation of a Savonius rotor using the boundary element method

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Abstract

The Boundary Element Method (BEM) is a very effective technique for solving complex domain and moving boundary problems, because it only involves calculations on the boundary thus reducing the processing time. For nonlinear and time dependent problems the BEM method presents some domain integrals that can be transformed into equivalent boundary integrals by the dual reciprocity approach applied in a subdomain basis; this method is known as Multi Domain Dual Reciprocity MD-DRM. In this paper some flow examples with moving boundaries will be solved using the MD-DRM method to predict particle trajectories and in particular the simulation of a Savonius rotor with a more complex geometry will be carried out to predict the velocity field, tractions and torque in this equipment.

Keywords: moving boundaries, tracking particles, savonius rotor, mixing.

1 Introduction

The recent energy crisis has required the world to look in the direction of alternative energy sources, wind power being the most widely used, mainly in coast line and offshore applications with large scale horizontal axis turbines. Even though these arrangements have high power outputs and efficiencies, they require large, low populated areas, along with high initial investments. In light of this, the urban generation market has shown to be a good possibility of development. But in urban and suburban areas, wind is not as reliable, with low wind speeds and fluctuating directions due to the interaction with the surrounding constructions. This panorama does not allow for the use of horizontal axis turbines but is well suited for a vertical axis turbine such as the Savonius rotor.



Although this turbines have been in the market for several years, different situations have slowed their development, and, in today's world, the experimental design is not an option due to high costs involved and the slow rate of improvement achievable. The obvious answer is to develop a computational method that can accurately predict the behaviour of a given configuration under an specific set of conditions, but there are certain difficulties involved in this process.

The first factor that must be taken into account is that the flow field around the Savonius turbine is not constant as it rotates, because different parts of the configuration face the flow at each angular position. Secondly, its operation characteristics create a high index of vorticity around the configuration, requiring a strong numerical method that can evaluate the flow under such extreme conditions.

Conventional numerical methods for fluid dynamics problems such as finite volumes and finite elements are not very efficient because of excessive computational time and high memory requirements. On the other hand, the boundary element method BEM is faster and more reliable because it reduces all the calculations to the boundary of the problem or at least to the boundary of each domain element (cell) in which the original region is divided [12] rendering this method more suitable for moving boundary problems.

In order to apply the BEM method to complex flow problems that involve inertia terms and gravity, it is necessary to find a fundamental solution or Green function for the partial differential equations that govern the phenomenon. However, since it is not possible to know such fundamental solution for a general model, it is required to group all of the non-linear and non-homogeneous terms into a pseudo-body force which leads to some domain integrals that might be evaluated by cell integration [1] which consumes high computer time and it makes the BEM lose its boundary only character. Therefore a few methods have been developed to transform the domain integrals into equivalent boundary integrals such as the dual reciprocity method (DRM) [11] that has greatly benefited from recent advances in multidimensional interpolation with radial functions (RBF) [10]. The first comparison of application between the usual conical function $1+r$ and the augmented thin plate splines as well as report of improvement of accuracy in the DRM approach when subdomains are used was reported by Popov and Power [12].

The greatest problem with DRM is that it requires many multiplications of fully populated matrices and for complex non-linear problems it needs a large number of nodes for the interpolation of the non-homogeneous terms. In order to solve these problems and limitations it is usual to use domain decomposition together with DRM, in which the original domain is divided into subregions and in each of them the full integral formula is applied, this method is known as Multidomain Dual Reciprocity (MD-DRM) [13, 2]. In the MD-DRM matrices are sparse like the finite element method and hence the solution is fast and efficient, and although there is an internal mesh and domains cell integration is avoided altogether.

When simulating rotating machines such as a Savonius turbine the labor intensity is amplified by the additional complexity of moving boundaries. There are two kinds of moving boundaries: those related with free surfaces in fluids, and those involving moving solid boundaries, this latter case will be analysed in this paper.



An additional advantage of using the BEM is that it gives us as a direct result the tractions on the moving boundaries, allowing for the calculation of the torque in this equipment, which in terms for the design of a Savonius rotor.

The purpose of this research is to present a simulation that can accurately deal with a moving boundary of a Savonius rotor based on the BEM technique. Several examples will be solved to show the versatility the numerical method for complex moving boundaries, and also to compare the performance of the traditional single domain BEM with the MD-DRM when particle tracking is concerned.

2 Mathematical model

2.1 Governing equations

The system of mass and momentum conservation equations for the unsteady state flow of an isothermal fluid are given in tensor notation by

$$\frac{\partial u_i}{\partial x_i} = 0, \quad x \in \Omega \quad (1)$$

$$-\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j}, \quad x \in \Omega \quad (2)$$

with boundary conditions

$$u_i = u_{i0}, \quad x \in \Gamma_u \quad (3)$$

$$t_i = \sigma_{ij} n_j = t_{i0}, \quad x \in \Gamma_t \quad (4)$$

where σ_{ij} is the total stress tensor; u_i is the velocity vector; t_i is the traction vector; n_i is the outward unit normal to the boundary $\Gamma = \Gamma_u + \Gamma_t$ of the volume Ω and p the pressure. The thermophysical properties are: ρ the fluid density, μ the viscosity.

The total stress tensor is obtained by, $\sigma_{ij} = -p\delta_{ij} + \mu\varepsilon_{ij}$, with $\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ being the strain rate tensor.

For the Savonius turbine, the torque T can be calculated from the traction field on the rotor's boundary by integrating along the boundary, the product $\vec{R} \times \vec{t}$, where $\vec{R}$ is the position vector of each point.

2.2 Multi-domain integral formulation

In the multi-domain BEM approach, the original domain is divided into smaller subregions or domain elements, each of them enclosed by a certain number of boundary elements. The solution of the partial differential equations Eq. (1)-(2) in each subdomain is given in terms of the corresponding integral representation formula. Also, adjacent domain-elements have to be matched according to some continuity or compatibility conditions for all the variables namely velocity and traction [3].



From the integral representation of Eq. (2) given in [7], it is found that the i component of the velocity field at a point x of the n -th subregion bounded by the contour Γ_n that encloses the sub-domain Ω_n is given by

$$c_{ik}u_i(x) = \int_{\Gamma_n} K_{kj}(x, y)u_j(y)d\Gamma_y - \int_{\Gamma_n} U_i^k(x, y)t_i(y)d\Gamma_y \\ - \int_{\Omega_n} U_i^k(x, y)g_i(y)d\Omega_y \quad (5)$$

for $n = 1, 2, \dots, M$ where M is the total number of subregions.

In the above equations $t_i(y)$ is the i component of the traction due to the flow field $(\vec{u}, p)$. The kernel $U_i^k(x, y)$ is the fundamental singular solution of the Newtonian Stokes system of equations (Stokeslet), where r is the distance from point y to any other point x , i.e. $r = |x - y|$, and $K_{ik}(x, y)$ is the traction corresponding to the flow field $(\vec{u}^k(x, y), q^k(x, y))$. The coefficients c_{ik} have values between δ_{ik} and 0, being equal to $\frac{1}{2}\delta_{ik}$ for smooth boundaries, and equal to $c_{ik} = \delta_{ik}$ for points inside the domain Ω_n .

The final set of equations is completed by assembling the integral equations Eq. (5) for each domain element, using the traction equilibrium and velocity compatibility at the common interfaces between subregions, i.e. $u_i^{(+)} = u_i^{(-)}$ and $\sigma_{ij}^{(+)}n_j + \sigma_{ij}^{(-)}n_j = 0$.

The domain integrals in expression (5) account for all the nonhomogeneous terms in the original partial differential equations (2). Where g_i stands for the term $\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j}$, and all together can be converted into equivalent boundary integrals in each subdomain by the Multi-Domain Dual Reciprocity or MD-DRM.

To express the domain integral in Eq. (5) in terms of equivalent boundary integrals, the Dual Reciprocity Method (DRM) approximation is introduced. The basic idea is to expand the $\vec{g}(x)$ term using radial interpolation functions at each sub-region.

An augmented spline consists of a radial basis function (RBF) plus a series of additional global functions[6]. In this particular case, it will be considered that there are $A = 3$ augmentation global functions from the set $\{1, x_1, x_2\}$. The RBF used in this work is the thin-plate spline. Although many different RBFs are discussed in the literature [6], the thin plate spline has features of optimality and convergence that make it the recommended choice [9].

The expansion of the $\vec{g}(x)$ term, when applied to the $N + L$ collocation nodes, with N being the number of nodes on the boundary and L the number of internal nodes, will generate $2(N + L)$ linear equations with $2(N + L + A)$ unknowns and therefore $2A$ additional conditions are necessary which basically guarantee an optimum interpolant [6].

Now, it is necessary to define an auxiliary velocity field $(\hat{U}_i^{lm}(x), \hat{p}^{lm}(x))$ [14], as obtained in previous works [9, 3]. At this point we can apply Green's identities to the auxiliary velocity field to obtain integral representation formulae only in terms of boundary integrals [14, 11, 2, 5], thus



$$\begin{aligned}
& c_{kj}(x)u_k(x) - \int_{\Gamma_n} K_{kj}(x,y)u_j(y)d\Gamma_y + \int_{\Gamma_n} U_i^k(x,y)t_j(y)d\Gamma_y \\
& = \sum_{m=1}^{N+L+A} \alpha_l^m \left\{ c_{kj}(x)\hat{U}_k^{lm}(x) - \int_{\Gamma_n} K_{kj}(x,y)\hat{U}_j^{lm}(y)d\Gamma_y \right. \\
& \quad \left. + \int_{\Gamma_n} U_i^k(x,y)\hat{t}_j^{lm}(y)d\Gamma_y \right\} \quad (6)
\end{aligned}$$

3 Discretization of the integral equations

For the numerical solution of the problem the surface Γ_n of each subregion or domain element can be discretised by means of isoparametric linear boundary elements. Along each element the integrals are calculated in terms of the nodal values of the velocity and tractions and using linear interpolation functions. In this way Eq.(6) can be written in matrix from as:

$$(cu)_i - \mathbf{H}_{ik}\mathbf{u}_k + \mathbf{G}_{ik}\mathbf{t}_k^N = \sum_{m=1}^{N+P} \alpha_j (\mathbf{H}_{ik}\hat{\mathbf{t}}_{kj} - \mathbf{G}_{ik}\hat{\mathbf{u}}_{kj} + (c\hat{\mathbf{u}}_j)) \quad (7)$$

where $\mathbf{H}_{ik}$ and $\mathbf{G}_{ik}$ are the standard influence matrices resulting from the integrations over the boundary elements, the index i represents de collocation nodes, k the nodes at the integration elements and j the DRM collocation points. In the present case, these matrices result from the integration over the four elements that form each subdomain.

To find the value of the coefficients α_l^m in Eq.(7), we expand the $\vec{g}(x)$ term using radial interpolation functions at each of the N chosen collocation nodes. In this way the following systems of equations are obtained,

$$\mathbf{F}\alpha = \mathbf{g} \quad (8)$$

that can be inverted to get the values of the unknown coefficients. When the nodal points are all distinct, the matrix resulting from any of the radial basis function interpolation given before is always non singular [8].

When the discrete integral equations for each domain-element Eq.(7) are put together, the final systems of equations are written in matrix notation as,

$$\mathbf{H}\mathbf{u} - \mathbf{G}\mathbf{t} = -(\mathbf{H}\hat{\mathbf{u}} - \mathbf{G}\hat{\mathbf{t}})\alpha \quad (9)$$

As shown in [4] the vector α is expressed in terms of velocities at the collocation nodes, and Eq. (9) is a nonlinear system that can be solved using the Newton-Raphson scheme combined with a line-search algorithm intended to reduce the error at each iteration.

Because the term g in 5 includes time derivatives of the velocity, the most straightforward procedure to approximate them was finite differences:



$$\frac{\partial u_i}{\partial t} \approx \frac{u_i^{(k+1)} - u_i^{(k)}}{\Delta t} \quad (10)$$

Since the boundaries rotate, domain geometry changes at each time step, and a new mesh would have to be generated at each instant. To avoid the calculations involved in the remeshing, the original mesh is rotated around the turbine's axis an angular step in accordance with the time step and the angular velocity for which the velocity field is being calculated. In order to obtain the time derivative (10) it is necessary to calculate the velocity of each node of the mesh at the previous instant. Knowing that the positions of the nodes changed from the last time step, a multidimensional interpolation with radial functions and the collocation method on each of the previous positions of the mesh nodes is employed.

Once the velocity field is calculated, the velocities of a different set of points contained in the general domain are obtained by a second multidimensional interpolation with radial functions using each of the mesh nodes for the collocation method. The previous procedure is done globally performing the collocation on the entire set of mesh nodes, and redone at each step. After obtaining the velocity by interpolation, the displacement of each point is calculated by a simple Euler method and the new position of each point will be generated for the next time step and new configuration of the mesh.

It must be noted that the boundary element method is ideally suited to track particles in the flow since the solution at any internal location can be obtained quite easily from the geometry matrices that define the radial interpolation (8). Numerically tracking tracer points through out the rotation of the Savonius turbine allows the user not only to visualize the deformation of tracer lines, in different regions but also to calculate strain rates and tractions at every time step because the tracer points carry information of velocity and velocity gradients.

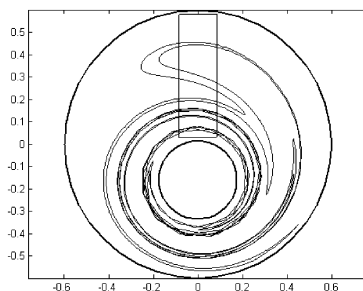


Figure 1: Excentric cylinder mixer. Numerical results.



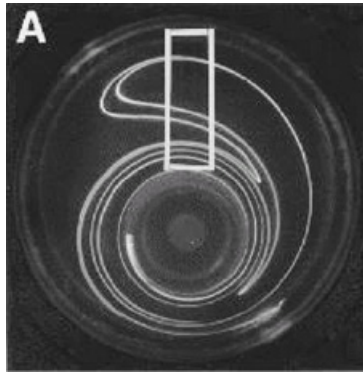


Figure 2: Excentric cylinder mixer. Experimental results.

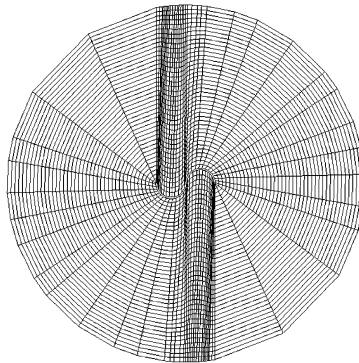


Figure 3: Multidomain quadrilateral mesh.

3.1 Numerical results

To test the effectiveness of the proposed numerical method for the analysis of rotatory configurations, an excentric cylinder mixer was simulated, and the results were compared with an experimental study developed by Professor James Ottino. The numerical results obtained by the MD-DRM method are shown in figure 1, while the experimental results by professor Ottino can be seen in figure 2 (from website of Prof. J.M. Ottino, Dep. of Chemical Engineering, Northwestern U.), and from the comparison it is evident the high accuracy of the numerical method for simulation of complex fluid configurations. Each domain element is made up of four boundary elements. The mixing protocol for the experiment involves a counter-clockwise rotation of the inner cylinder throughout 1080° , followed by a clockwise rotation of the outer cylinder throughout 240° , and finally a repetition of the same movement in the inner cylinder.

The second numerical example is a Savonius rotor. The meshed general domain

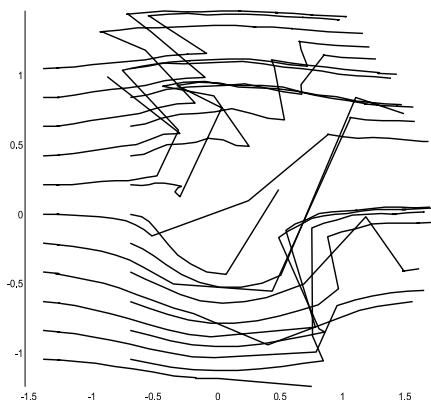


Figure 4: Tracer point trajectories.

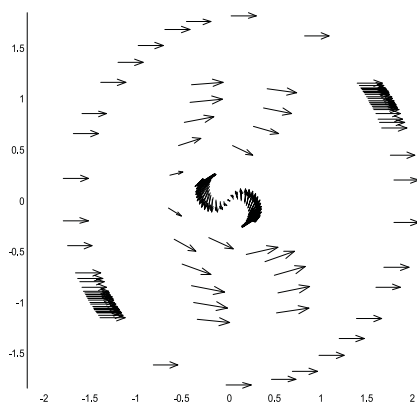


Figure 5: Velocity field.

and the geometry are shown in figure 3. The mesh is structured and it has a higher density near the rotor. A 4.5 m/s wind speed and a 0.8 wing tip speed ratio (ratio of the wing tip tangential speed to wind speed) was used in the analysis as it corresponds to an actual practical situation; 22 tracer point were used and their trajectories are shown in figure 4. The velocity field for specific position is shown in figure 5. The torque of the configuration is shown in figure 6. In it, a high variation of the torque can be seen as the turbine rotates. Another important aspect is the symmetry of the profile relative to a 180° position. Both situations go along with the physical characteristics of the rotor.

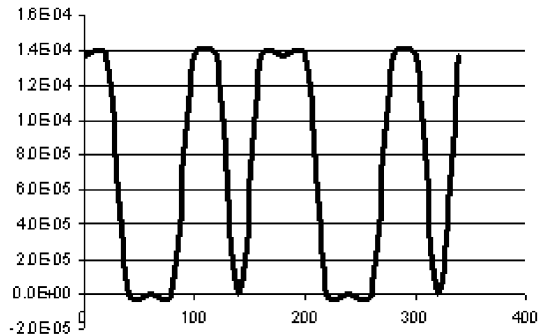


Figure 6: Torque history.

3.2 Conclusions

The MD-DRM was modified to model particle trajectories in a complex flow problem with rotating elements. The updating of positions and instant velocities of each tracer point was conducted by multidimensional interpolation with radial functions and an Euler method. The results obtained by this method agree with the experimental results and the expected physical characteristics of complex rotatory machinery.

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