

Matrix compression schemes for wavelet BEM and their performance

K. Koro & K. Abe

*Department of Civil Engineering and Architecture, Niigata University,
Japan*

Abstract

The influence of the truncation schemes for wavelet BEM on the compression rates of the coefficient matrices is investigated through the asymptotic estimation and numerical experiments. In the present paper the truncation schemes given by either Schneider's level-dependent scheme or the Beylkin-type algorithm are considered. The theoretical estimation of the number of stored entries shows $O(N^{1+\alpha})$ ($0 < \alpha < 1$) for the Beylkin-type compression and $O(N(\log N)^\beta)$ ($\beta \geq 1$) for the level-dependent truncation. However, in actual BE analysis the Beylkin-type algorithm enables us to compress the coefficient matrix at the rates comparable to those for Schneider's scheme.

Keywords: wavelet BEM, compression scheme, large-scale problem.

1 Introduction

The wavelet BEM, in which a boundary integral equation is discretized with the wavelets, enables us to reduce the computational cost of boundary element (BE) analysis. The cost reduction results from the compression of coefficient matrices of the discretized equation. The matrix compression is achieved by the truncation of small matrix entries. The truncation process in the wavelet BEM consists of the following two steps: (i) the determination of a threshold and (ii) the selection of truncated entries. The truncation is executed for entries smaller than the threshold value. The choice of the truncation scheme is hence an important issue to attain the high-performance computing in the wavelet BEM.

The matrix compression algorithms have been discussed by e.g. Beylkin *et al* [1] and Schneider [2]. In the Beylkin-type compression scheme a single threshold is applied to every matrix entry. The level of the threshold in practical analysis



can be determined [3] such that the asymptotic convergence rate of BE solution is retained. This compression scheme leads to the number of non-zero entries of $O(N^{1+\alpha})$ ($0 < \alpha < 1$, N : degree of freedom) [3]. On the other hand, Schneider's compression algorithm is one of the reliable level-dependent schemes. The threshold varies with the resolution level of wavelets; it is controlled using several truncation parameters. This scheme ensures to retain the convergence rate if the parameters are consistent with the prescribed conditions shown in the present paper. The number of non-zero entries can be reduced to $O(N(\log N)^\beta)$ ($\beta \geq 1$) [2].

The theoretical estimation of the number of non-zero entries is a basis of the widely-accepted wisdom that the level-dependent truncation scheme as Schneider's algorithm is of advantage in cost reduction. However, the theoretical estimation cannot predict the actual number of stored entries. This paper discusses the performance of the compression schemes in terms of the number of non-zero entries and clarifies their practicability.

2 Boundary element formulation using wavelets

Let us consider the wavelet BEM for Laplace problems. The boundary integral equation for Laplace problem is described as

$$c(\mathbf{x})u(\mathbf{x}) + \int_{\Gamma} q^*(\mathbf{x}, \mathbf{y})u(\mathbf{y})d\Gamma_y = \int_{\Gamma} u^*(\mathbf{x}, \mathbf{y})q(\mathbf{y})d\Gamma_y + f(\mathbf{x}), \quad (1)$$

where Γ denotes the boundary, and $\mathbf{x}, \mathbf{y} \in \Gamma$. u and u^* are the potential and the corresponding fundamental solution, respectively. $q = \partial u / \partial n$, $q^* = \partial u^* / \partial n$ and n is the outward normal direction at $\mathbf{y}$. The function $f(\mathbf{x})$ is defined as either $f = 0$ (internal problems) or $f = U^\infty(\mathbf{x})$ (external problems). $U^\infty(\mathbf{x})$ gives the distribution of the potential at infinity, and $c(\mathbf{x})$ is the free term.

We next substitute the approximation $\tilde{u}$ and $\tilde{q}$ into the true solution u and q . In wavelet BEM $\tilde{u}$ and $\tilde{q}$ are defined by the following wavelet series:

$$\tilde{u} = \sum_{l=1}^{n_s} \hat{u}_{0,l} \phi_{0,l} + \sum_{k=0}^m \sum_{l=1}^{n_k} \tilde{u}_{k,l} \psi_{k,l}, \quad \tilde{q} = \sum_{l=1}^{n_s} \hat{q}_{0,l} \phi_{0,l} + \sum_{k=0}^m \sum_{l=1}^{n_k} \tilde{q}_{k,l} \psi_{k,l}, \quad (2)$$

where $\phi_{0,l}$ and $\psi_{k,l}$ are the scaling function and the wavelets, respectively. These functions have been presented in e.g., Refs. [4, 5]. $\hat{u}_{0,l}$, $\tilde{u}_{k,l}$, $\hat{q}_{0,l}$ and $\tilde{q}_{k,l}$ are the expansion coefficients, and m is the finest resolution level.

The discretized equation of eqn 1 is derived with the Galerkin method. The resulting algebraic equation is described as follows:

$$\mathbf{H}\mathbf{u} = \mathbf{G}\mathbf{q} + \mathbf{f}, \quad (3)$$

where the vectors $\mathbf{u}$ and $\mathbf{q}$ ($\in \mathbb{R}^N$, N : degree of freedom) correspond to the expansion coefficients of $\tilde{u}$ and $\tilde{q}$. $\mathbf{G}$ and $\mathbf{H}$ ($\in \mathbb{R}^{N \times N}$) are the coefficient matrices, and



their entries g_{ij} and h_{ij} ($i, j = 1, 2, \dots, N$) are given by

$$g_{ij} = \int_{\Gamma} w_i \int_{\Gamma} u^* w_j d\Gamma^2, \quad h_{ij} = \frac{1}{2} \int_{\Gamma} w_i w_j d\Gamma + \int_{\Gamma} w_i \int_{\Gamma} q^* w_j d\Gamma^2, \quad (4)$$

where the basis functions w_i and w_j stand for either $\phi_{0,l}$ or $\psi_{k,l}$. The vector $\mathbf{f}$ has the components f_i ($i = 1, 2, \dots, N$) defined as the inner product of w_i and f on Γ .

3 Beylkin-type truncation scheme

In the Beylkin-type scheme the truncated entries are selected by comparing the matrix entries g_{ij} and h_{ij} with a single threshold τ . The matrix entries satisfying the following criterion are replaced with null entries without calculating double integration 4:

$$\bar{g}_{ij} < \tau \cdot G_{max}, \quad \bar{h}_{ij} < \tau \cdot H_{max}. \quad (5)$$

In the truncation process the entries with $\bar{r} \leq \nu(2^{-k_i} + 2^{-k_j})$ are stored even if eqn 5 holds where $\bar{r} = \text{dist}(\text{supp } w_i, \text{supp } w_j)$, and k_i and k_j are the resolution level of the basis w_i and w_j . G_{max} and H_{max} are the maximum values of g_{ij} and h_{ij} , respectively. In the present paper the threshold τ is set to $\tau = \alpha' N^{-\beta'}$ ($\alpha', \beta' > 0$) by the semi-analytical technique [3]. The parameters α' and β' depend on the order of integral operator, the problem dimensionality and the boundary conditions; however, these are independent of the level k_i and k_j , i.e., the threshold is level-independent. $\bar{g}_{ij}$ and $\bar{h}_{ij}$ are the Taylor approximation of the entries. These are introduced to avoid the unnecessary integration for the truncated entries.

The number of non-zero entries of the matrices $\mathbf{G}$ and $\mathbf{H}$ are estimated via the following two steps: (i) estimating the number of stored entries in a submatrix specified with the levels k_i and k_j , and (ii) summing them with respect to both k_i and k_j . This estimation is based on the approaches presented in Refs [2, 3, 6].

Designating a coefficient matrix $\mathbf{A}$, we can describe the number of stored entries $\mathcal{N}(\mathbf{A})$ as follows:

$$\mathcal{N}(\mathbf{A}) \leq C \sum_{k_i=0}^m \sum_{k_j=0}^m \mathcal{N}(\mathbf{A}_{k_i, k_j}), \quad (6)$$

where $\mathbf{A}_{k_i, k_j}$ is the submatrix corresponding to the level (k_i, k_j) .

In eqn 6 $\mathcal{N}(\mathbf{A}_{k_i, k_j})$ is estimated as

$$\mathcal{N}(\mathbf{A}_{k_i, k_j}) \leq C \left(2^{k_i s} + 2^{k_j s} + \gamma 2^{(k_i + k_j) s} \delta^s \right), \quad (7)$$

where $\gamma > 0$, and $s = 1$ (2-D problems) or $s = 2$ (3-D problems). In the Beylkin-type scheme the threshold δ is derived from truncation condition 5 as follows:

$$\delta = \begin{cases} \bar{\delta}, & \bar{\delta} = C \tau^{-\frac{1}{\mu}} 2^{-\lambda \frac{(k_i + k_j)}{2}}, & (r_0 < \bar{\delta}) \\ r_0, & r_0 = \nu(2^{-k_i} + 2^{-k_j}), & (r_0 \geq \bar{\delta}), \end{cases} \quad (8)$$



where $\mu = s + 2n$ and $\lambda = 1$ for the matrix $\mathbf{H}$, while $\mu = s + 2n - 1$ and $\lambda = (s + 2n)/\mu$ for $\mathbf{G}$. n is the order of vanishing moments.

As shown in eqn 8, the different value of δ is chosen according to the resolution level. The submatrix $\mathbf{A}_{k_i, k_j}$ with $r_0 < \bar{\delta}$, namely $\delta = \bar{\delta}$, exists in blocks satisfying

$$k_i - \omega < k_j < k_i + \omega, \quad (k_i = 1, 2, \dots, m), \quad (9)$$

where the band width ω is defined as $2^{\omega/2} = C' \nu^{-1} \tau^{-1/\mu}$.

We now substitute eqn 7 into eqn 6, and calculate the resulting summation in consideration of eqns 8 and 9. The number of non-zero entries for the Beylkin-type compression can be consequently estimated as follows:

$$\begin{aligned} \mathcal{N}(\mathbf{A}) &\leq C_1 \sum_{k_i=0}^m \sum_{k_j=0}^m (2^{k_i s} + 2^{k_j s}) + C_2 \tau^{-\frac{s}{\mu}} \sum_{k_i=0}^{m-\omega} \sum_{k_j=k_i}^{k_i+\omega} 2^{\frac{\varepsilon s}{2}(k_i+k_j)} \\ &\quad + C_3 \tau^{-\frac{s}{\mu}} \sum_{k_i=m-\omega+1}^m \sum_{k_j=k_i}^m 2^{\frac{\varepsilon s}{2}(k_i+k_j)} \quad (10) \\ &\leq C_1(m+1)2^{(m+1)s} + C_2 2^{(m+1)s} + C_3 2^{(1+\alpha)(m+1)s} \\ &\leq C_1 N \log N + C_2 N + C_3 N^{1+\alpha}, \end{aligned}$$

where $(m+1) \approx \log N$ and $2^{(m+1)s} \approx N$. In eqn 10, $\varepsilon = (s + 2n - 2)/\mu$ and $\alpha = (s\beta' - 1)/\mu$ for the matrix $\mathbf{G}$, while $\varepsilon = 1$ and $\alpha = s\beta'/\mu$ for the matrix $\mathbf{H}$. We thus obtain the asymptotic estimation of the number of non-zero entries; $\mathcal{N}(\mathbf{A}) = O(N^{1+\alpha})$ ($0 < \alpha < 1$).

4 Schneider's level-dependent scheme

The Schneider's truncation scheme selects the truncated entries according to the distance $\bar{r}$. In this scheme the matrix entries with $\bar{r} > \delta_{k_i, k_j}$ are neglected without the calculation of integration 4. The threshold δ_{k_i, k_j} has the following level-dependent form [2]:

$$\delta_{k_i, k_j} = \begin{cases} a \cdot \max \left\{ 2^{-\min(k_i, k_j)}, (m+1)^{\frac{1}{2n+r}} 2^{(m+1)-(k_i+k_j)} \right\}, & (p+1 = n+r), \\ a \cdot \max \left\{ 2^{-\min(k_i, k_j)}, 2^{\frac{(m+1)(2p'-r)-(k_i+k_j)(n+p')}{2n+r}} \right\}, & (p+1 < n+r), \end{cases} \quad (11)$$

where $a > 1$, $p+1 < p' < n+r$, and p is the order of polynomials in the basis. r is the order of the corresponding integral operator, and hence $r = 0$ (matrix $\mathbf{H}$) and $r = -1$ (matrix $\mathbf{G}$).



The number of non-zero entries is estimated through the similar approach stated in the previous section. Considering eqn 11, we obtain the following asymptotic estimation:

$$\begin{aligned}\mathcal{N}(\mathbf{A}) &\leq C'_1(m+1)2^{(m+1)s} + C'_2 2^{(m+1)s} (m+1)^{2+\frac{s}{2n+r}} \\ &\leq C'_1 N \log N + C'_2 N (\log N)^{2+\frac{s}{2n+r}},\end{aligned}\quad (12)$$

for $p+1 = n+r$. Hence, the number of non-zero entries has $\mathcal{N}(\mathbf{A}) = O(N(\log N)^{2+\frac{s}{2n+r}})$. When the discretization condition satisfies $p+1 < n+r$, we have

$$\begin{aligned}\mathcal{N}(\mathbf{A}) &\leq C'_3(m+1)2^{(m+1)s} + C'_4 2^{(m+1)s} \\ &\quad + C'_5 2^{(m+1)s} 2^{-\frac{2(n+r-p')^2}{(2n+r)(n+p')}} (m+1)s + C'_6 2^{(m+1)s} 2^{-\frac{n+r-p'}{n+p'}} (m+1)s, \quad (13) \\ &\leq C'_3 N \log N + C'_4 N + C'_5 N^{1-\frac{2(n+r-p')^2}{(2n+r)(n+p')}} + C'_6 N^{1-\frac{n+r-p'}{n+p'}}.\end{aligned}$$

In eqn 13,

$$1 - \frac{2(n+r-p')^2}{(2n+r)(n+p')} < 1, \quad 1 - \frac{n+r-p'}{n+p'} < 1, \quad (14)$$

and hence $\mathcal{N}(\mathbf{A})$ shows the asymptotic behaviour of $O(N \log N)$ for $p+1 < n+r$.

The matrix compression based on criterion 11 leads to quasi-optimal complexity. The Schneider's original algorithm has the second compression process only for $p+1 < n+r$. Although this process is not implemented in the present paper, the combination of the present level-dependent algorithm and the second compression enables us to reduce the computational work to $O(N)$ [2].

5 Numerical results

As stated above the Beylkin-type truncation ensures the number of stored entries of $O(N^{1+\alpha})$ ($0 < \alpha < 1$), while the Schneider's level-dependent compression achieves $O(N(\log N)^\beta)$ ($\beta \geq 1$) quasi-optimal complexity. The theoretical estimation on the number of non-zero entries is a basis of the widely accepted wisdom that higher compression rates of the coefficient matrices are gained by the level-dependent truncation. However, the actual number of the entries cannot be predicted by the estimation.

In the present paper we discuss the practicability of the matrix compression schemes. The discussion is based on the actual number of entries, which number is obtained by numerical tests. The test examples are a 2-D mixed boundary value problem (see Figure 1(a)) and a 3-D external Neumann problem (see Figure 1(b)). In the numerical tests the univariable wavelet for 2-D analysis was



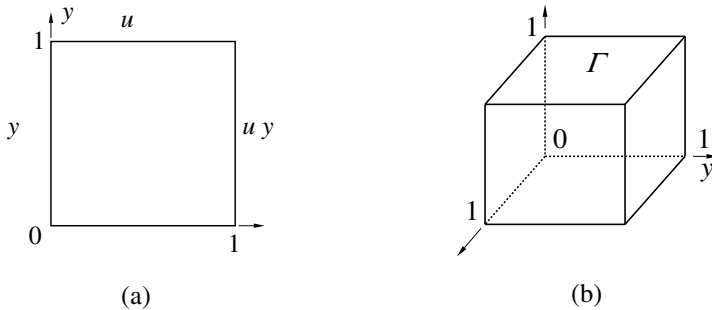


Figure 1: Test examples. (a): 2-D mixed boundary value problem. (b): 3-D external Neumann problem. $q = 0$ (on Γ), $U^\infty = x$.

given by the piecewise constant non-orthogonal wavelet [4] with third-order vanishing moments. For 3-D problem the orthonormal wavelet with first-order vanishing moment was employed for the surface wavelet. This wavelet is constructed from piecewise constant functions with triangular supports.

Figure 2 shows the number of non-zero entries in the coefficient matrix for 2-D problem. Note that the figure corresponds to the matrix in conjunction with the unknown vector. This is because the other coefficient matrix, corresponding to the known vector, is not stored. For the 2-D problem shown in Figure 1(a), the threshold δ_{k_i, k_j} used for Schneider's level-dependent truncation algorithm is given by the second equation in eqn 11. The parameter p' can be chosen from $1 < p' < 2$. The other parameter a is prescribed as $a = 1$. The Beylkin-type algorithm compresses the coefficient matrix at comparable rates to that for Schneider's scheme, in spite of the theoretical estimation of $O(N^{1+\alpha})$. The number of non-zero entries for $N < 10^5$ is rather less than the alternative scheme. The difference between these two results is decreased in progression of the degree of freedom, and nothing is the difference for $N \approx 10^5$. The degree of freedom of $N \approx 10^5$ may be a limit that numerical tests can be undertaken in the usual computer environment. We thus conclude that for 2-D problem the performance of the Beylkin-type scheme is comparable to Schneider's level-dependent truncation in practical wavelet-based BE analysis. The present theoretical estimation stated in the previous section shows $O(N \log N)$ for the level-dependent scheme and $O(N^{1.13})$ (matrix $\mathbf{G}$) and $O(N^{1.26})$ (matrix $\mathbf{H}$) for the Beylkin-type algorithm. The numerical results validates this estimation for 2-D problem.

The results for 3-D external Neumann problem are shown in Figure 3. In this analysis we use piecewise constant surface wavelet with first-order vanishing moment. The threshold in Schneider's algorithm is restricted to the first condition in eqn 11. This reasonable setting of the threshold corresponds to the results labelled "Case 1" in Figure 3. The term "Case 2" indicates that the corresponding numerical result is obtained using an illegal threshold of the second equation in

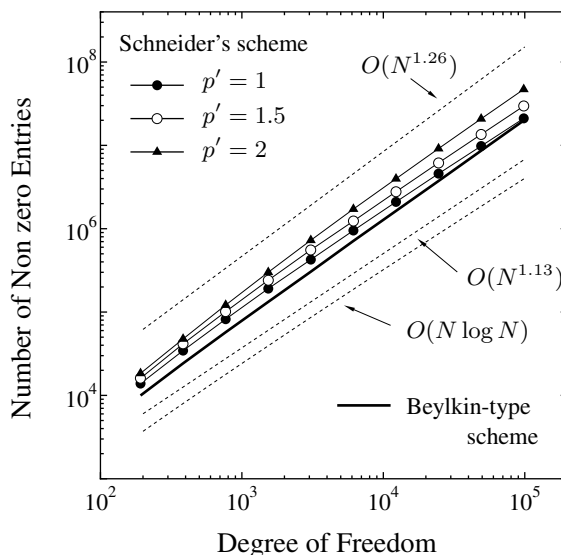


Figure 2: The number of non-zero entries of coefficient matrix corresponding to the known vector. The test example is the 2-D problem shown in Figure 1(a).

eqn 11. For the reasonable setting the implementation of Schneider's algorithm results in lower compression rates than that of Beylkin-type scheme. This is consistent with the conclusion for 2-D analysis. As shown in Figure 3, the truncation with Case 2 provides the highest compression rate among the three approaches. The illegal setting may however, cause the reduction of accuracy of BE solution. Hence, we should choose the reasonable threshold to obtain reliable results with the wavelet BEM, when the matrix compression is executed based on Schneider's level-dependent scheme. The Beylkin-type truncation achieves the number of non-zero entries of $O(N^{1.44})$ in the present numerical test. The theoretical estimation of $O(N^{1.40})$ is thus validated from these results.

6 Conclusions

The performance of two truncation schemes for the wavelet BEM has been assessed based on the actual number of stored entries of the coefficient matrices. The truncation algorithms have been constructed with the Beylkin-type scheme and Schneider's level-dependent approach. The theoretical estimation of the number of non-zero entries gives $O(N^{1+\alpha})$ ($0 < \alpha < 1$) for the Beylkin-type truncation. The numerical results have proven this estimate. The non-optimal order for the Beylkin-type matrix compression is theoretically inferior to $O(N(\log N)^\beta)$ ($\beta \geq 1$) for Schneider's level-dependent truncation scheme. The actual computa-



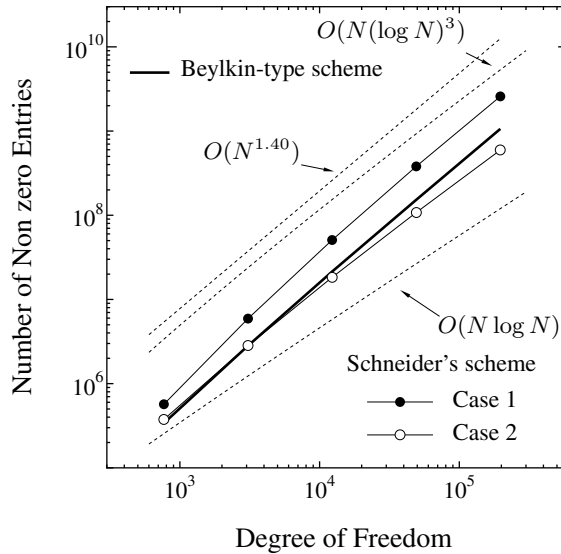


Figure 3: The number of non-zero entries of the matrix $\mathbf{H}$. The test example is the 3-D problem shown in Figure 1(b).

tional performance is however, comparable to that for the level-dependent scheme within the practical scale of BE analysis.

References

- [1] Beylkin, G., Coifman, R., Rokhlin, V.: Fast wavelet transforms and numerical algorithms I. *Comm. Pure Appl. Math.*, **44**, pp.141–183, 1991.
- [2] Schneider, R.: Multiskalen- und Wavelet-Matrixkompression: Analysis-basierte Methoden zur effizienten Lösung großer vollbesetzter Gleichungssysteme. B.G.Teubner, Stuttgart, 1998.
- [3] Koro, K., Abe, K.: A practical determination strategy of optimal threshold parameter for matrix compression in wavelet BEM. *Int. J. Numer. Meth. Engng.*, **57**, pp.169–191, 2003.
- [4] Koro, K., Abe, K.: Non-orthogonal spline wavelets for boundary element analysis. *Engng. Anal. Bound. Elems.*, **25**, pp.149–164, 2001.
- [5] von Petersdorff, T., Schwab, C. & Schneider, R.: Multiwavelets for second-kind integral equations. *SIAM J. Numer. Anal.*, **34**(6), pp.2212–2227, 1997.
- [6] Lage, C., Schwab, C.: Wavelet Galerkin algorithms for boundary integral equations. *SIAM J. Sci. Comput.*, **20**(6), pp.2195–2222, 1999.

