Sensitivity analysis of the Poisson equation using the Trefftz method

E. Kita[1], Y. Ikeda[2] & N. Kamiya[1]
[1] *Graduate School of Information Sciences, Nagoya University, Nagoya 464-8301, Japan*
[2] *Department of Mechanical Engineering, Daidoh Institute of Technology, Nagoya 457-8530, Japan*

Abstract

A sensitivity analysis scheme for the boundary value problem of the three-dimensional Poisson equation by using the Trefftz method is described in this paper.

In the present scheme, the inhomogeneous term of the three-dimensional Poisson equation is approximated with the polynomial function in Cartesian coordinates to derive the related particular solution. The use of the particular solution transforms the boundary value problem of the Poisson equation into that of the Laplace equation. Since the unknown parameters included into the particular solution depend on the unknown function, the derived boundary value problem is solved by the iterative process.

The function is approximated with the superposition of the T-complete functions and the particular solution. Therefore, direct differentiation of the solution leads to the sensitivities. The boundary-specified value is taken as the variable for the sensitivity analysis and then the sensitivity analysis formulations are described.

1 Introduction

Sensitivity analysis schemes are very important tools for solving inverse problems. Many researchers have presented the sensitivity analysis schemes based on the finite and boundary element methods [1, 2, 3, 4]. They, however, have some difficulties. So, authors have presented the sensitivity analysis schemes based on Trefftz method for boundary value problems of Laplace equation and two-dimensional elastostatic problems [5, 6, 7]. In this paper, the sensitivity analysis schemes based



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on the Trefftz method are developed for the boundary value problems of the three-dimensional Poisson equations.

In the present scheme, an inhomogeneous term of the three-dimensional Poisson equation is approximated with the polynomial function in Cartesian coordinates to determine the particular solution related to the inhomogeneous term of the governing equation. The use of the particular solution transforms the boundary value problem of the Poisson equation into that of Laplace equation. The solution of the transformed problem can be estimated by the help of the ordinary Trefftz method for the boundary value problem of the Laplace equation and the iterative process. The unknown function (solution) of the original problem is approximated with the particular solution and the homogeneous solutions (so-called T-complete functions) obtained from boundary value problem of the Laplace equation. Therefore, direct differentiation of the solution leads to the sensitivities. The boundary-specified value is taken as the variable for the sensitivity analysis and then, the sensitivity analysis formulations are described when the inhomogeneous term depends on the unknown function alone and both the unknown function and its derivatives.

2 Trefftz formulation for three-dimensional Laplace equation

In the present formulation, the boundary value problem of the Poisson equation is transformed into that of the Laplace equation. So, the Trefftz formulation for the Laplace equation is described briefly in this section.

2.1 Collocation-type formulation

The governing equation and the boundary conditions of the three-dimensional Poisson equation are given as follows:

$$\nabla^2 u = 0 \ \ (\text{in } \Omega) \tag{1}$$

and

$$\begin{aligned} u &= \bar{u} \quad (\text{on } \Gamma_1) \\ q \left(\equiv \frac{\partial u}{\partial n} \right) &= \bar{q} \quad (\text{on } \Gamma_2) \end{aligned} \tag{2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Besides, Ω, Γ_1 and Γ_2 denote the object domain under consideration, the potential (u) specified and the flux (q) specified boundaries, respectively.

In the Trefftz formulation, the solution of the problem is approximated with the superposition of the regular T-complete functions satisfying the governing equation.



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2.2 T-complete function

The T-complete function for the three-dimensional Poisson equation is derived from the variable separation of the governing equation in the spherical coordinates [8].

$$\begin{aligned} \boldsymbol{u}^* &= \left\{ r^\nu \sqrt{\tfrac{(2\nu+1)!(\nu-\mu)!}{4\pi(\nu+\mu)!}} P_\nu^\mu(\cos\theta) e^{j\mu\phi} \right\} \\ &\quad (\nu = 0, 1, 2, \cdots; -\nu - 1 < \mu < \nu + 1) \end{aligned} \tag{3}$$

where r, θ and ϕ are the system of the spherical coordinates and j the imaginary unit. Besides, P_ν^μ denotes the associated Legendre polynomial function. According to the ideas of Herrera [9, 10] and Zielinski and Zienkiewicz [11], one employs as the T-complete function the following function obtained by eliminating the constant coefficient from equation (3);

$$\begin{aligned} \boldsymbol{u}^* &= \left\{ r^\nu P_\nu^\mu(\cos\theta) e^{j\mu\phi} \right\} \\ &\quad (\nu = 0, 1, 2, \cdots; -\nu - 1 < \mu < \nu + 1) \end{aligned} \tag{4}$$

Besides, the T-complete function related is given as follows

$$\begin{aligned} \boldsymbol{q}^* \equiv \frac{\partial \boldsymbol{u}^*}{\partial n} = \big\{ & \nu r^{\nu-1} P_\nu^\mu(\cos\theta) e^{j\mu\phi} n_r \\ & + r^\nu \frac{\partial P_\nu^\mu(\cos\theta)}{\partial\theta} e^{j\mu\phi} n_\theta \\ & + j\mu r^\nu P_\nu^\mu(\cos\theta) e^{j\mu\phi} n_\phi \big\} \end{aligned} \tag{5}$$

where

$$\frac{\partial P_\nu^\mu(\cos\theta)}{\partial\theta} = \frac{\begin{array}{c}(\nu-\mu+1) P_{\nu+1}^\mu(\cos\theta) \\ -(\nu+1)\cos\theta P_\nu^\mu(\cos\theta)\end{array}}{\sin\theta} \tag{6}$$

n_r, n_θ and n_ϕ denote the r, θ and ϕ-directional components of the normal unit vector, respectively.

2.3 Collocation-type formulation

If the T-complete functions for Eq.(1) is referred as u_l^*, the solution u is approximated as follows.

$$\begin{aligned} u \simeq \tilde{u} &= a_1 u_1^* + a_2 u_2^* + \cdots + a_N u_N^* \\ &\equiv \boldsymbol{a}^T \boldsymbol{u}^* \end{aligned} \tag{7}$$

where $\boldsymbol{a} = \{a_1, a_2, \cdots, a_N\}^T$ is the unknown parameter vector and $(\tilde{})$ the approximate solution. Besides, N is the total number of the T-complete functions. Direct

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differentiation of the above expression in the normal direction on the boundary leads to the flux expression;

$$q \simeq \tilde{q}\left(\equiv \frac{\partial \tilde{u}}{\partial n}\right) = a_1 q_1^* + a_2 q_2^* + \cdots + a_N q_N^* \equiv \boldsymbol{a}^T \boldsymbol{q}^* \tag{8}$$

Equations (7) and (8) satisfy the governing equation. However, they do not satisfy the boundary conditions. Substitution of Eqs.(7) and (8) to (2) leads to the residual expressions;

$$\begin{aligned} R_1 &\equiv \tilde{u} - \bar{u} = \boldsymbol{a}^T \boldsymbol{u}^* - \bar{u} \neq 0 \quad \text{on } \Gamma_1 \\ R_2 &\equiv \tilde{q} - \bar{q} = \boldsymbol{a}^T \boldsymbol{q}^* - \bar{q} \neq 0 \quad \text{on } \Gamma_2 \end{aligned} \tag{9}$$

In the collocation-type Trefftz method, the unknown parameter vector $\boldsymbol{a}$ is determined so that the residuals are force to be zero on the boundary collocation points P_m. From Eq.(9), we have

$$\begin{aligned} R_1(P_m) &= \boldsymbol{a}^T \boldsymbol{u}^*(P_m) - \bar{u}(P_m) = 0 \\ &\quad (P_m \text{ on } \Gamma_1, m = 1 \ldots, M_1) \\ R_2(P_m) &= \boldsymbol{a}^T \boldsymbol{q}^*(P_m) - \bar{q}(P_m) = 0 \\ &\quad (P_m \text{ on } \Gamma_2, m = 1 \ldots, M_2) \end{aligned} \tag{10}$$

where M_1 and M_2 denote the total numbers of the collocation points on the boundary Γ_1 and Γ_2, respectively.

$$\begin{aligned} \boldsymbol{a}^T \boldsymbol{u}^*(P_m) &= \bar{u}(P_m) \quad (P_m \text{ on } \Gamma_1) \\ \boldsymbol{a}^T \boldsymbol{q}^*(P_m) &= \bar{q}(P_m) \quad (P_m \text{ on } \Gamma_2) \end{aligned} \tag{11}$$

By rearranging the above equations in the matrix form, we have

$$\begin{bmatrix} u_{11}^* & u_{12}^* & \cdots & u_{1N}^* \\ u_{21}^* & u_{22}^* & \cdots & u_{2N}^* \\ \vdots & \vdots & \ddots & \vdots \\ u_{M_1 1}^* & u_{M_1 2}^* & \cdots & u_{M_1 N}^* \\ q_{11}^* & q_{12}^* & \cdots & q_{1N}^* \\ q_{21}^* & q_{22}^* & \cdots & q_{2N}^* \\ \vdots & \vdots & \ddots & \vdots \\ q_{M_2 1}^* & q_{M_2 2}^* & \cdots & q_{M_2 N}^* \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_{M_1} \\ \bar{q}_1 \\ \bar{q}_2 \\ \vdots \\ \bar{q}_{M_2} \end{bmatrix} \tag{12}$$

where

$$u_l^*(P_m) \equiv u_{ml}^*$$

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$$q_l^*(P_m) \equiv q_{ml}^*$$
$$\bar{u}(P_m) \equiv \bar{u}_m$$
$$\bar{q}(P_m) \equiv \bar{q}_m$$

Finally, we have

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f} \tag{13}$$

Since equation (13) is an over-defined system of equations, it is solved with the singular value decomposition of the LAPACK software [12].

3 Trefftz formulation for boundary value problem of three-dimensional Poisson equation

3.1 Boundary value problem of three-dimensional Poisson equation

The governing equation and the boundary conditions of the problem are given as

$$\nabla^2 u + b(x, y, z, u) = 0 \ \ (\text{in } \Omega) \tag{14}$$

and

$$\left.\begin{array}{rcll} u & = & \bar{u} & (\text{on } \Gamma_1) \\ q & = & \bar{q} & (\text{on } \Gamma_2) \end{array}\right\} \tag{15}$$

3.2 Variable transformation of governing equation

The inhomogeneous term b is approximated withe the polynomial functions in the global orthogonal coordinates as follows.

$$\begin{aligned} b &= c_1 + c_2 x + c_3 y + c_4 z + \cdots \\ &= \boldsymbol{c}^T \boldsymbol{r} \end{aligned} \tag{16}$$

where $\boldsymbol{c}$ and $\boldsymbol{r}$ denote the unknown parameter vector and the vector of the terms of the polynomial function, respectively.

$$\boldsymbol{c}^T = \{c_1, c_2, \cdots, c_{20}\} \tag{17}$$

$$\begin{aligned} \boldsymbol{r}^T = \{&1, x, y, z, x^2, y^2, z^2, xy, xz, yz, \\ &x^3, y^3, z^3, x^2 y, x^2 z, xy^2, \\ &xz^2, xyz, y^2 z, yz^2, \cdots\} \end{aligned} \tag{18}$$

Note that the parameter vector $\boldsymbol{c}$ depends on the unknown function u. The use of equation (16) transforms the governing equation (14) as follows.

$$\nabla^2 u + \boldsymbol{c}^T \boldsymbol{r} = 0 \tag{19}$$

We shall refer the homogeneous solution of equation (19) as u^h and the particular solution related to the term r_i as u_i^p. The solution of the problem u is approximated

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as follows.

$$u = u^h + \boldsymbol{c}^T \boldsymbol{u}^p \tag{20}$$

Since the elements of the vector $\boldsymbol{r}$ are the term of the polynomial function, the related particular solution u_i^p can be determined easily. The use of the equation (20) transforms the governing equation (14) and the boundary condition (15) as follows:

$$\nabla^2 u^h = 0 \ \ (\text{in } \Omega) \tag{21}$$

and

$$\left.\begin{array}{rcll} u^h & = & \bar{u} - \boldsymbol{c}^T \boldsymbol{u}^p & (\text{on } \Gamma_1) \\ q^h & = & \bar{q} - \boldsymbol{c}^T \boldsymbol{q}^p & (\text{on } \Gamma_2) \end{array}\right\} \tag{22}$$

Since the unknown parameter vector $\boldsymbol{c}$ depends on the unknown function u, the vector $\boldsymbol{c}$ is determined by the iterative scheme. The iterative process is shown in the next subsection.

3.3 Iterative process for updating $\boldsymbol{c}$

Holding equation (16) at the iterative step (k) and $(k+1)$, we have

$$b(x, y, z, u)^{(k+1)} = \boldsymbol{r}^T \boldsymbol{c}^{(k+1)}$$
$$b(x, y, z, u)^{(k)} = \boldsymbol{r}^T \boldsymbol{c}^{(k)}$$

Subtracting each side of the above equations, we have

$$\begin{aligned} b^{(k+1)} - b^{(k)} &= \boldsymbol{r}^T (\boldsymbol{c}^{(k+1)} - \boldsymbol{c}^{(k)}) \\ &\equiv \boldsymbol{r}^T \Delta \boldsymbol{c} \end{aligned} \tag{23}$$

The vector $\Delta \boldsymbol{c}$ is determined so that the equation (23) is held at the collocation points taken on the boundary and within the domain, which are referred as "computing points". Holding the equation (23) at the computing point Q_i, we have

$$\begin{aligned} \boldsymbol{r}^T(Q_i)\Delta \boldsymbol{c} &= b^{(k+1)} - b^{(k)} \\ &= b(u(Q_i^{(k)})) - \boldsymbol{r}^T \boldsymbol{c}^{(k)} \\ &\equiv \Delta b(Q_i) \end{aligned} \tag{24}$$

$b^{(k)}$ means the approximate solution before solving the boundary value problem and therefore, it is estimated as follows

$$b^{(k)} = \boldsymbol{r}^T \boldsymbol{c}^{(k)}$$

On the other hand, $b^{(k)}$ is estimated from the solution of the boundary value problem$u(Q_i^{(k)})$ as follows

$$b^{(k+1)} = b(u(Q_i^{(k)})).$$

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holding equation (24) at all computing points Q_i, we have

$$\boldsymbol{D}\Delta\boldsymbol{c} = \boldsymbol{f} \tag{25}$$

where $\boldsymbol{D}$ and $\boldsymbol{f}$ denote the coefficient matrix of the terms of the polynomial function and the unknown parameter vector, respectively. Equation (25) is solved for $\Delta\boldsymbol{c}$ by the singular value decomposition [12] and the parameters are updated by

$$\boldsymbol{c}^{(k+1)} = \boldsymbol{c}^{(k)} + \Delta\boldsymbol{c} \tag{26}$$

The convergence criterion is defined as

$$\eta \equiv \frac{1}{M_c}\sum_{i=1}^{M_c} |\Delta b(Q_i)| < \eta_c \tag{27}$$

where η_c is positive constant specified by a user in advance.

3.4 Solution algorithm

The algorithm of the present method is as follows.

1. Initial k as $k \leftarrow 0$ and assume the initial value of the parameter $\boldsymbol{c}^0$.
2. Solve the boundary value problem defined by equations (21) and (22) and estimate the inhomogeneous terms at the computing points.
3. Estimate the convergence criterion. If the criterion is satisfied, the results are printed out. If not so, the process goes to the next.
4. Solve the equation (25) for $\Delta\boldsymbol{c}$.
5. Update $\boldsymbol{c}$ and increment and increase k by 1
6. Go to the step 2.

4 Sensitivity analysis scheme

4.1 In the case of $b = b(u)$

First we shall consider that an inhomogeneous term b depends only on an unknown function u. Specified values on boundary are considered as the variable for sensitivity analysis to explain the present formulation.

Substituting Eq.(7) to Eq.(20) leads to

$$u = u^h + \boldsymbol{c}^T\boldsymbol{u}^p = \boldsymbol{a}^T\boldsymbol{u}^* + \boldsymbol{c}^T\boldsymbol{u}^p \tag{28}$$

Differentiating both sides of Eq.(28) with respect to the boundary-specified value, we have

$$\dot{u} = \dot{\boldsymbol{a}}^T\boldsymbol{u}^* + \dot{\boldsymbol{c}}^T\boldsymbol{u}^p \tag{29}$$

Holding Eq.(29) at all computing points and rearranging in the matrix form, we have

$$\boldsymbol{K}_1\dot{\boldsymbol{a}} + \boldsymbol{B}_1\dot{\boldsymbol{c}} - \boldsymbol{I}\dot{\boldsymbol{u}} = 0 \tag{30}$$

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Besides, direct differentiation of Eq.(13) leads to

$$\boldsymbol{K}\dot{\boldsymbol{a}} + \boldsymbol{B}\dot{\boldsymbol{c}} = \dot{\boldsymbol{f}} \tag{31}$$

Finally, direct differentiation of Eq.(16) leads to

$$\dot{\boldsymbol{c}}^T\boldsymbol{r} - \frac{\partial b}{\partial u}\dot{u} = 0 \tag{32}$$

Holding the above equation at all computing points and rearranging in the matrix form, we have

$$\boldsymbol{G}\dot{\boldsymbol{c}} - \boldsymbol{H}\dot{\boldsymbol{u}} = 0 \tag{33}$$

Rearranging Eqs.(30), (31) and (33) in the matrix form, we have

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{K}_1 & \boldsymbol{B}_1 & -\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{G} & -\boldsymbol{H} \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{a}} \\ \dot{\boldsymbol{c}} \\ \dot{\boldsymbol{u}} \end{Bmatrix} = \begin{Bmatrix} \dot{\boldsymbol{f}} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{Bmatrix} \tag{34}$$

Equation (34) is solved for $\dot{\boldsymbol{u}}$ to determine the sensitivities.

4.2 In the case of $b = b(u, u_{,j})$

We shall consider in the case of the inhomogeneous term depending first-derivative of $b = b(u, u_{,j})$, where $(\ )_{,j}$ denotes the first-order partial differentiation with respect to j. Specified values on boundary are considered as the variable of sensitivity analysis to explain the formulation.

Direct differentiation of Eq.(16) with respect to the boundary-specified value leads to

$$\dot{\boldsymbol{c}}^T\boldsymbol{r} - \frac{\partial b}{\partial u}\dot{u} - \frac{\partial b}{\partial u_{,j}}\dot{u}_{,j} = 0$$

Holding the above equation at all collocation points, we have

$$\boldsymbol{G}\dot{\boldsymbol{c}} - \boldsymbol{H}_1\dot{\boldsymbol{u}} - \boldsymbol{H}_2\dot{\boldsymbol{u}}_{,j} = 0 \tag{35}$$

Besides, differentiating Eq.(28) with respect to j, we have

$$u_{,j} = \boldsymbol{a}^T\boldsymbol{u}^*_{,j} + \boldsymbol{c}^T\boldsymbol{u}^p_{,j} \tag{36}$$

Direct differentiation of $u_{,j}$ with respect to the boundary-specified value leads to

$$\dot{u}_{,j} = \dot{\boldsymbol{a}}^T\boldsymbol{u}^*_{,j} + \dot{\boldsymbol{c}}^T\boldsymbol{u}^p_{,j} \tag{37}$$

Holding the above equation at each computing point and rearranging them in the matrix form, we have

$$\boldsymbol{K}_2\dot{\boldsymbol{a}} + \boldsymbol{B}_2\dot{\boldsymbol{c}} - \boldsymbol{I}\dot{\boldsymbol{u}}_{,j} = 0 \tag{38}$$



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Rearranging Eq.(29), (30), (35) and (38), we have

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{K}_1 & \boldsymbol{B}_1 & -\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{K}_2 & \boldsymbol{B}_2 & \boldsymbol{0} & -\boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{G} & -\boldsymbol{H}_1 & -\boldsymbol{H}_2 \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{a}} \\ \dot{\boldsymbol{c}} \\ \dot{\boldsymbol{u}} \\ \dot{\boldsymbol{u}}_{,j} \end{Bmatrix} = \begin{Bmatrix} \dot{\boldsymbol{f}} \\ \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{Bmatrix} \tag{39}$$

The equation is solved for $\dot{\boldsymbol{u}}$ to determine the sensitivities.

5 Conclusions

This paper describes the sensitivity analysis formulation for the boundary value problem of the three-dimensional Poisson equation by the help of the Trefftz method.

In the present scheme, an inhomogeneous term of the three-dimensional Poisson equation is approximated with the polynomial function in Cartesian coordinates to determine the particular solution related to the inhomogeneous term of the governing equation. The unknown function (solution) of the original problem is approximated with the superposition of the derived particular solution and the T-complete functions related to the Laplace equation. Therefore, direct differentiation of the solution leads to the sensitivities. The boundary-specified value is taken as the variable for the sensitivity analysis and then, the sensitivity analysis formulations are described when the inhomogeneous term depends on the unknown function alone and both the unknown function and its derivatives.

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