Wave and water motion analysis using combined boundary element and weighted finite difference method

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Abstract

In an earlier study, we identified the combined boundary element and weighted finite difference method (WFDM) used to analyze water flow (Kanoh et al. [1]). In order to apply the combined method in our problem the analytical region is divided into four sub-domains. (1) The first and third are irrotational flow domains that exist on both sides of a submerged breakwater, in which the moving boundary technique and the intricate boundary conditions are managed by the boundary-element method (Kanoh et al. [2]). (2) The third is a turbulent flow domain that is located around the breakwater, in which the WFDM provides an accurate solution for the wave and water motion through the openings of the breakwater. (3) The fourth is an extra sub-domain where we set the imaginary boundary and define the passing-through condition of waves. To reproduce the moving free surface in WFDM solutions, first the WFDM grids are changed in the vertical position. However, the velocities of WFDM solutions tend to be diverse in the region near the piston wave-maker. To decrease the divergence, we try to move the WFDM grids both horizontally and vertically. Referring to the velocity vectors of the water motion visualized with experimental laboratory models, the effect and accuracy of the alternative hybrid method are estimated. Keywords: combined method, wave and water motion, weighted finite difference method, experimental laboratory model, passing-through condition of wave.

1 Introduction

Regarding the internal boundary condition through the openings on the crown plank and the sidewalls of a perforated submerged breakwater, we succeeded in
expressing the internal flow conditions by using the newly developed WFDM. If an imaginary boundary is set in our analytical domain, it is expected that the area of the analytical region for the problem is decreased and computational time is saved. For this purpose, the imaginary boundary should represent the passing-through condition of wave and water motions. The precise technique to define the passing-through condition is described below.

2 Governing equations

The Navier-Stokes, continuous and Laplace equations are used to describe two-dimensional flows in the vertical $(x_1, x_2)$ plane as illustrated in Figure 1.

2.1 Equations for irrotational flow

The Laplace equation for the time-dependent velocity potential $\phi(x_1, x_2, t)$ can be used in the irrotational flow region $\Omega_{irro}$ as explained previously

$$\nabla^2 \phi = 0 \quad \text{in } \Omega_{irro}, \quad (1)$$

The relation of the time derivative of $\phi$, velocities $u_1$ and $u_2$ and pressure $P$ is described in the irrotational flow region $\Omega_{irro}$ as

$$\frac{\partial \phi}{\partial t} - 1/2(u_1^2 + u_2^2) - gx_2 - P / \rho = 0 \quad \text{in } \Omega_{irro}, \quad (2)$$

where $t$ is time, $g$ is the gravity acceleration, $x_2$ is the vertical direction and $\rho$ is the density.

Figure 1: Separation and boundaries of wave and water motion regions.
2.2 Equations for rotational and irrotational flow

The continuous and Navier-Stokes equations govern the incompressible flow in the rotational and irrotational flow regions ($\Omega_{ro}$, $\Omega_{pass}$). In the vertical ($x_1$, $x_2$) plane, these equations are shown as follows:

\[
\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0 , \tag{3}
\]

\[
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} = -\frac{1}{\rho} \frac{\partial P}{\partial x_1} + \nu_e \left( \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} \right) , \tag{4.1}
\]

\[
\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} = -\frac{1}{\rho} \frac{\partial P}{\partial x_2} + g + \nu_e \left( \frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} \right) , \tag{4.2}
\]

\[
\nu_e = \nu_m + \nu_t , \tag{5}
\]

where $x_1$, $x_2$ are the horizontal and vertical directions, $u_1$ and $u_2$ describe the velocities of the $x_1$ and $x_2$ directions, $g$ is the gravity acceleration and $\nu_e$, $\nu_m$ and $\nu_t$ are the kinematic, molecular, and time-dependent viscosity coefficients, respectively.

2.3 Viscosity coefficient for turbulent flow

In turbulent flow regions, the kinematic viscosity varies with the location and the time. To reproduce the turbulent flow numerically, it is important to give the kinematic viscosity appropriately. Some numerical models are used for the purpose, such as k-$\varepsilon$ and k-$\omega$ models (Abe et al. [3], Peng et al. [4]). In this paper, we estimate the viscosity with reference to our previous work (Kanoh [5]), in which, by using the order estimation method, we proposed the expression of the viscosity as

\[
\nu_e = k V^2 \quad \text{or} \quad \nu = \alpha V^2 \Delta t , \tag{6}
\]

where $V$ is the velocity vector at the position and the time under consideration, $k$ or $\alpha$ is a coefficient and $\Delta t$ is the time increment. The value of the coefficient $k$ or $\alpha$ is changed and estimated such that the WFDM solutions show good agreement with the visualized velocity-vectors obtained in our experiment since we consider that $k$ or $\alpha$ may change its value if the flow fields and conditions vary. The velocity $V$ is given as

\[
V = (u_1^2 + u_2^2)^{1/2} , \tag{7}
\]

where $u_1$ and $u_2$ are the solutions of WFDM calculated $\Delta t$ time before. We consider that the expression of Eqs. (6) and (7) can define the viscosity $\nu_e$ in the turbulent flow since the velocities $u_1$ and $u_2$ are accurately obtained at every second and every place in the flow region and the coefficient $k$ or $\alpha$ is estimated appropriately.
3 Development of our coupled method

3.1 Flow regions and sub-domains

We deal with wave and water motion over and through a perforated submerged breakwater as shown in Figure 1 and try to develop and apply the proposed combined BEM and WFDM for the problem. The permeable quadrilateral breakwater has several openings on its vertical walls and horizontal crown. The specifications of our experimental model are almost same as those described in our previous paper (Kanoh et al. [2]). The analytical domain is divided into three sub-domains. The first is an irrotational flow domain ($\Omega_{\text{irro}}$), in which the moving boundary technique and the intricate boundary conditions are managed by our BEM (Kanoh et al. [2]). The second is a turbulent or rotational flow domain ($\Omega_{\text{ro}}$), in which the developed WFDM provides an accurate solution for the wave and water motion through the openings of the perforated submerged breakwater. The third is a domain ($\Omega_{\text{pass}}$) where we define the passing-through condition of wave and water motion.

3.2 Passing-through condition

Regarding the passing-through condition of waves, in order to decrease the area of the analytical region and the computational time, we propose an alternative methodology such that the weighted finite difference method provides the normal derivative of the velocity $\partial u/\partial n$ at the boundary in the extra sub-domain ($\Omega_{\text{pass}}$). WFDM can be used to define and calculate the derivative of the velocity since the governing equations for WFDM include the velocity $u$. However, BEM does not easily provide the normal derivative of the velocity because the governing equations for BEM do not include the velocity but the potential $\phi$. The reason described above is why, in the irrotational flow region, we set the third domain ($\Omega_{\text{pass}}$) where WFDM provides the passing-through condition of wave and water motion.

3.3 Moving-boundary condition of WFDM

As the free surface changes its position, even in the turbulent region, WFDM is developed to handle the moving boundary. To reproduce the moving free surface in WFDM solutions, first the WFDM grids are changed in the vertical position since we consider that the trapezoidal grids may express the shape of the free surface of the wave. Namely, as WFDM can be used not only for rectangular but also quadrilateral grids, the ordinate values of the grids are moved according to the length $u_v \Delta t$, where $u_v$ describes the vertical velocity of wave motions and $\Delta t$ is the time increment. However, the velocities of WFDM solutions tend to diverge in the region near the interface $\Gamma_{\text{ro-irro}}$ and the piston wave-maker. To decrease the divergence, next we try to move the WFDM grids both horizontally and vertically.
3.4 Boundary conditions

The analytical domain contains several kinds of boundaries as illustrated in Figure 1. We have previously proposed the free-surface conditions for WFDM in the rotational flow region, the boundary conditions between the rotational and irrotational flow regions (Kanoh et al. [1], [2]). We also have defined other boundary conditions for the wave-generator boundary $\Gamma_{\text{gen}}$, the bottom and impermeable breakwater surfaces $\Gamma_{\text{b}}$ and other fixed boundaries, the horizontal crown plank $\Gamma_{\text{C}}$ or the vertical wall boundary $\Gamma_{\text{W}}$ of a submerged breakwater (Kanoh et al. [2]). Some boundary conditions and their applications for our coupling technique are newly derived and described below.

3.5 Coupling technique

As the analytical domain is divided into four sub-domains, it is necessary to define the technique for coupling BEM with WFDM along three interfaces between each two regions. We introduce the coupling techniques and investigate the effects below.

3.5.1 BEM sub-domain for coupling

The coupling technique using the BEM sub-domain is described here. Referring to Figure 2(a), we first let the BEM sub-domains coincide with the WFDM grids that intrude on the BEM region by one column. The normal derivative of the potential $\partial \phi / \partial n$ is calculated by the BEM on each boundary element of the sub-domains. We transform these to the velocities on the WFDM grids using the expression as

$$u_1 = - \frac{\partial \phi}{\partial x_1}, \quad u_2 = - \frac{\partial \phi}{\partial x_2}. \quad (8)$$

Then the transformed velocities are given on the WFDM grids as the known boundary values. However, near the interface, the velocities transformed from BEM solutions tend to be diverse. When we set BEM sub-domains along the interface $\Gamma_{\text{WFDM-BEM}}$, too much stiffness will be yielded for the problem.

In order to avoid the too much stiffness for the problem, we use BEM internal points without sub-domains along the interface.

3.5.2 BEM internal point without sub-domain for coupling

In order to avoid the too much stiffness for the problem, we use BEM internal points without sub-domains along the interface.

3.5.2.1 Right interface $\Gamma_{\text{WFDM-BEM}}$

The potential $\phi$ is calculated by BEM on each internal point of Figure 2(b). Along the right interface ($\Gamma_{\text{WFDM-BEM}}$), we calculate the velocities from the potential values on the WFDM grids using the finite differential expression as

$$u_1 = (\phi_{\bigcirc} - \phi_{\bullet})/\Delta x_1, \quad u_2 = (\phi_{\bigtriangleup} - \phi_{\bigotimes})/\Delta x_2, \quad (9)$$

where $\phi_{\bigcirc}, \phi_{\bullet}, \phi_{\bigtriangleup}, \phi_{\bigotimes}$ are the potential values on the corresponding points ($\bigcirc, \bullet, \bigtriangleup, \bigotimes$) in Figure 2(b). Here the WFDM grids also intrude on the BEM region by one column. No divergence appeared in the BEM and WFDM solutions in
this coupling technique, so we adopted the technique to couple BEM with WFDM along the interface $\Gamma_{WFDM-BEM}$.

Figure 2(a): BEM sub-domains coincide with WFDM grids. (b): BEM internal points (○) in WFDM grids.

Figure 3: Without BEM sub-domains and internal points.

Figure 4: Finite difference for passing-through condition ($1_{Le} u_n = 1_{Ri} u_n$).
3.5.2.2 Middle interface $\Gamma_{\text{BEM-WFDM}}$

The WFDM grids do not intrude on the BEM region, and no internal points are set along the middle interface ($\Gamma_{\text{BEM-WFDM}}$) as shown in Figure 3. Using the velocities $u_\alpha$ and Eq. (10), we can put the normal derivative of potential $\partial \phi_n / \partial n$ on the boundary $\Gamma_{\text{BEM-WFDM}}$ as the Neumann condition:

$$\frac{\partial \phi_n}{\partial n} = -u_n \quad \text{on} \quad \Gamma_{\text{BEM-WFDM}},$$

where the velocity $u_n$ is obtained by WFDM. Once the normal derivative of $\phi$ is given on the boundary, the potential $\phi$ can be calculated by using our BEM in the irrotational flow region $\Omega_{\text{irro}}$.

3.6 Finite difference expression for passing-through condition

There are two boundaries ($\Gamma_{\text{imag}}$, $\Gamma_{\text{WFDM-BEM}}$) and two regions ($\Omega_{\text{WFDM}}$, $\Omega_{\text{BEM}}$) near the passing-through region as shown in Figure 4. Along the interface $\Gamma_{\text{WFDM-BEM}}$ we can use the same coupling technique as the one described in Section 3.5.2. Along the boundary $\Gamma_{\text{imag}}$ (or $\Gamma_{\text{pass}}$), we define the passing-through condition as follows: (1) The WFDM grids project out by one column as shown in Figure 4. (2) Let the velocity values on the both side grids of the boundary $\Gamma_{\text{imag}}$ equal each other. Namely, the relation $1_{\text{Le}} u_\alpha = 1_{\text{Ri}} u_\alpha$ is induced by the fact that the normal derivative of velocity equals 0 on the boundary $\Gamma_{\text{imag}}$. Applying this technique to the analysis of wave and water motion through the perforated breakwater, we confirmed that the wave passed through the imaginary boundary without changing its velocities and wave-heights. We consider that the passing-through condition is successfully completed.

4 WFDM and BEM analysis

4.1 WFDM model and scheme for rotational flows

By using WFDM, the convective diffusion and shallow water flow problems were accurately calculated. Referring to the WFDM model for the flow problems, we proposed the alternative WFDM model to analyze the two-dimensional rotational flows in the vertical $(x_1, x_2)$ plane (Kanoh et al. [1]). Hereby, $\Delta x_1 = \alpha h$, $\Delta x_2 = \beta h$ and $\Delta t$ are the lengths of the grids of the finite difference scheme in the WFDM model. The WFDM scheme for velocity $u_1$, which corresponds to the above WFDM model, can be written as

$$i,j u_1^{i+\Delta t} = u_1 W_{1\cdot} i,j u_1^i + u_1 W_{2\cdot} i+1,j u_1^i + u_1 W_{3\cdot} i+2,j u_1^i + F W_{1\cdot} i-1,j F^{t-\Delta t/2} + F W_{2\cdot} i,j F^{t-\Delta t/2} + F W_{3\cdot} i+1,j F^{t-\Delta t/2}$$

(11)
where \( u_{1}^{i+\Delta t} \) means the velocity in the \( x_1 \) direction on the point \((i\Delta x_1, j\Delta x_2)\) and at the time \((t+\Delta t)\). The moving-boundary technique can be applied to WFDM because the lengths of \( \Delta x_1 \) and \( \Delta x_2 \) change every second. Hereby, the weights \( u_1W_1, u_1W_2 \) and \( u_1W_3 \) are decided by solving the linear equations described in our previous paper (Kanoh et al. [1]). We consider that WFDM may be used to calculate accurately the wave and water motion in a geometrically and physically complicated flow field since the linear equations contain the Courant number (the dimension-less velocity), the diffusion number (the dimension-less viscosity) and the distortions of the grid. Namely, WFDM corresponds to the variations in velocity and viscosity and the complicated geometry of the flow field.

### 4.2 Boundary element expression

The discritized boundary element expression for analyzing the Laplace equation (Eq. (1)) is written as

\[
H\phi = Gq, \quad q = \frac{\partial \phi}{\partial n}, \quad H\phi' = Gq',
\]

where \( \phi' \) and \( q' \) are the time derivatives of \( \phi \) and \( q \), respectively (e.g. Kanoh et al. [2]).

### 5 Results and discussion

#### 5.1 BEM solutions and techniques

In analyzing the wave and water motion through the opening of the perforated submerged breakwaters by using BEM, the strong nonlinearity of the free-surface boundary condition should be solved. For this purpose, the B-spline technique was applied to restrain efficiently the oscillation of the free surface and make the shape very smooth. The newly proposed techniques for coupling BEM and WFDM were estimated numerically and determined in this paper. Using BEM solutions in the analysis above, we observed that the calculated transmission coefficients of the perforated submerged breakwaters showed good agreement with the experimental results obtained in our laboratory. Examining BEM solutions and the experimentally visualized velocities of the water motion as shown in Figure 5, we recognized that both lengths of the velocity vectors in BEM and the experimental results were comparable. On the other hand, there was a large difference in the directions of the velocity vectors. Namely, BEM solutions did not show the rotational movements but only those of the vertical or horizontal direction. However, we consider that BEM solutions may reproduce the eddy loss and the effective dissipation of the porous submerged breakwater since the lengths of velocity vectors of BEM solutions are comparable to those of the experimental results.
5.2 WFDM solutions and discussion

Referring to the numerical solutions and techniques previously and newly introduced by us, we obtained the following results: (1) With the analytical region of the problem divided into the irrotational, rotational and passing-through domains, BEM was applied to the first and WFDM was applied to the second and third domains. The boundary conditions on the boundaries between each two domains were newly estimated and defined. (2) We consider that WFDM model schemes are successfully developed to handle the moving boundary and analyze the rotational flows in wave motions against a submerged breakwater since WFDM solutions reproduce the rotational flow and free-surface movements in the problem. (3) Applying the passing-through condition to the analysis of wave and water motion through the perforated breakwater, we confirmed that the wave passed through the imaginary boundary without changing its velocities and wave heights. We consider that with WFDM and the extra domain, the passing-through condition is successfully set up. (4) WFDM has an advantage in that it is applicable to the upwind scheme and corresponds to the variations of velocity and viscosity and the complicated geometry of the flow field. Considering the facts estimated above, we expect the hybrid method to save computational time and yield good agreement with the experimental results of the wave and water motion through a porous submerged breakwater obtained in our laboratory.
6 Conclusion

We have developed the combined boundary element and weighted finite difference method to analyze the rotational and irrotational flows over and through a porous submerged breakwater. In order to reduce the area of the analytical region and save a large amount of computational time for the problem, the imaginary boundary in the extra domain and the passing-through technique for WFDM were introduced for the analysis of the problem. Using the experimental and numerical solutions, we attempted to set up the appropriate WFDM schemes and improve the hybrid method.

References


