Performance evaluation of some new time integration methods in elastodynamic problems formulated by dual reciprocity boundary element method

G. A. Velázquez Castillo & C. F. Loeffler
Mechanical Engineering Department, Universidade Federal do Espírito Santo, Brazil

Abstract

This paper presents the results obtained with Dual Reciprocity Boundary Element Formulation applied to dynamic problems. In spite of the generally good performance, wave propagation problems demand still greater attention in controlling the higher modes, especially under impact loading. To improve the traction response, numerical simulations were done with three time integration methods with parametric control of dissipation: the well-known Wilson-θ scheme, the HHT-α method and the recently devised CHL-β method.

1 Introduction

Dual Reciprocity (DR) is a technique of Boundary Element Method (BEM) initially idealized for the solution of free vibration problems [1]. DR allows obtaining the final matrix equations in a form similar to application of the Finite Element Method (FEM) with the difference that no domain integration is made. On the other hand, the time marching procedures with BEM requests the use of time integration method with some form of algorithmic damping. This is necessary to remove the participation of the spurious high-frequency modal components [2]. Due to the mixed formulation, displacements and tractions are calculated simultaneously and the influence of badly represented higher modes
can greatly affect the numerical response. The most common solution has been to use time integration method with fictitious damping in which the numerical dissipation control is done exclusively for the time step size. So, the accuracy of the numerical response must be warranted by the time integration method chosen. Houbolt's scheme has been used with relative success, especially with high order boundary elements [3], but it is still necessary to achieve a more effective procedure when constant elements were used.

This work presents results of two-dimensional scalar problems of impact loading. The choice of problems and the kind of solicitation are important to evaluate the performance of the proposed methods, because the representation of the response is rather difficult, especially the evaluation of tractions.

2 Basic equations

Consider the governing equation of the scalar wave in solids:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \ddot{u} \tag{1}$$

In equation (1), $u$ is the displacement and $c$ is the velocity of wave propagation. The domain $\Omega(X)$ is delimited by a boundary $\Gamma(X)$, where $X$ is a vector of the spatial variables. The following conditions are considered over the boundary:

$$u(X,t)=\bar{u}(X,t) \ on \ \Gamma_u(X) \ and \ \frac{\partial u(X,t)}{\partial n} = \frac{\partial \bar{u}(X,t)}{\partial n} \ on \ \Gamma_q(X) \tag{2}$$

$\Gamma_u(X)$ is the part of boundary where displacements are known and in a complementary way, $\Gamma_q(X)$ is the region of the boundary where surface tractions are prescribed. In equation (2); 'n' is the coordinate in the direction of the outward unit normal vector at a point on $\Gamma_q$. Initial conditions in $\Omega(x)$ are:

$$u(x,0)=u_0(x) \ and \ \ddot{u}(x,0)=\ddot{u}_0(x) \tag{3}$$

3 Basic BEM integral formulation

According to the most traditional BEM procedures, the governing equation must be written in the inverse integral form [4]. It can easily be obtained after performing some mathematical procedures on the left side of equation (1), where the Laplacian operator is applied. Thus, the following equation is achieved:

$$C(\xi) \ u(\xi) + \int_\Gamma q \ u^* d\Gamma - \int_\Gamma q \ u^* \ d\Gamma = -\frac{1}{c^2} \int_\Omega \ddot{u} \ u^* \ d\Omega \tag{4}$$

In equation (4) it was used the fundamental solution $u^*(\xi;x)$ as the auxiliary function. It is the solution in an infinite domain governed by the Poisson equation with a point source, represented by the Dirac delta function, at $X = \xi$. In two-dimensional problems:
where $r(\xi; X)$ is the Euclidian distance between source point $\xi$ and the field point $X$ in the domain $\Omega$. To achieve equation (4), the divergence's theorem and special properties of the Dirac delta function were used [4, 5].

4 Dual reciprocity procedure

The domain integral on the right-hand-side of equation (4) can be approximated through boundary integrals using DR's procedure. First, the displacement $\dot{u}$ is approximated by a sum of a finite number of new arbitrary functions as follows:

$$\dot{u}(X, t) = \dot{\alpha}^j(t)\psi^j_{,ii}(X), \quad j = 1, 2, \ldots, m$$

Consequently the domain integral can be written as:

$$\int_{\Omega} \dot{u} u^* d\Omega = \dot{\alpha}^j \int_{\Omega} \psi^j_{,ii} u^* d\Omega$$

The functions $\psi^j$ are arbitrary. The Euclidian distance between two points is an appropriate apation. Operations like those done to generate the inverse integral form in the left hand side of equation (4) can also be done in right hand side of equation (7), resulting in only boundary integral terms. The complete integral equation of motion is than obtained:

$$C(\xi)u(\xi) + \int_\Gamma u q^* d\Gamma - \int_\Gamma q u^* d\Gamma = [C(\xi)\psi^j(\xi) +$$

$$+ \int_\Gamma \psi^j q^* d\Gamma - \int_\Gamma \eta^j u^* d\Gamma] \frac{\dot{\alpha}(t)}{c^2}$$

where: $\eta^j = \psi^j_{,i} n_i = \frac{\partial \psi^j}{\partial n}$

The next step consists of dividing the boundary $\Gamma(x)$ into elements. In this work the discretization were restricted to constant elements. It is pointed out that besides the displacements $u$ and the surface tractions $q$ are considered constant along each element, the same happens with functions $\psi^j$ and $\eta^j$. It is done for the sake of simplicity, because they could be calculated exactly. So, equation (8) can be written for each boundary element generating a set of equations that, using matrix notation, can be written in the following way:

$$[M] \ddot{\hat{u}} + [H] [u] = [G] [q]$$

It was chosen a number of functions $\psi^j_{,ii}$ equal to the number of nodal points and the function $\alpha$ can be substituted and written in terms of $\ddot{\hat{u}}$, in agreement with the equation (6), that is:

$$[\ddot{\hat{\alpha}}] = [\psi^j_{,ii}]^{-1} [\ddot{\hat{u}}]$$
5 Time discretization

The time discretization is done using direct integration methods. These methods search for an appropriate relationship that allows the computation of future values from previous results in a consistent form. The approximated solution satisfies the equation of motion at discrete time points, \( t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, t_f \) [6]. Herein three approach operators were analyzed, whose algorithms will be shown in what follows.

5.1 Wilson-\( \theta \) algorithm

In this scheme it is assumed that the acceleration varies linearly in an extended time interval \( \Delta s = \theta \Delta t \). In FEM, values of \( \theta > 1.37 \) make the algorithm unconditionally stable. This scheme proposes [6]:

\[
\begin{align*}
\dot{u}_s &= \dot{u}_n + \Delta s \ddot{u}_n + \frac{\Delta s^2}{3} \dddot{u}_n + \frac{\Delta s^2}{6} \dddot{u}_s \\
\ddot{u}_s &= \ddot{u}_n + \Delta s \dddot{u}_n + \frac{\Delta s}{2} \dddot{u}_n + \frac{\Delta s}{2} \dddot{u}_s
\end{align*}
\] (12) (13)

5.2 CHL-\( \beta \) method

The CHL-\( \beta \) method is an algorithm that approximates the displacement and the velocity in the following way [7]:

\[
\begin{align*}
\ddot{u}_{n+1} &= \ddot{u}_n + \Delta t \left( \frac{1}{2} \dddot{u}_n + \beta \dddot{u}_{n+1} \right) \\
\dddot{u}_{n+1} &= \dddot{u}_n + \Delta t \left[ \frac{1}{2} \dddot{u}_n + \frac{3}{2} \dddot{u}_{n+1} \right]
\end{align*}
\] (14) (15)

For the FEM it is demonstrated that the algorithm is stable for \( 1 \leq \beta \leq 28/27 \) and as the value of \( \beta \) increases, so the system dissipation also increases. The simulations will show that for BEM the variation of \( \beta \) is much wider.

5.3 HHT-\( \alpha \) method

This method is an adaptation of the Newmark method according the following expressions [8]:

\[
\begin{align*}
\ddot{u}_{n+1} &= \ddot{u}_n + \Delta t \left( \dddot{u}_n + \frac{\Delta t}{2} \dddot{u}_{n+1} \right)
\end{align*}
\] (16)
\[ u_{n+1} = u_n + \Delta t \left[ (1 - \gamma) \dot{u}_n + \gamma \ddot{u}_{n+1} \right] \]  

(17)

For the FEM it can be shown that, for \(-1/3 \leq \alpha \leq 0\), \(\gamma = (1 - 2\alpha)/2\) and \(\beta = (1 - \alpha)^2/4\), the algorithm becomes unconditionally stable.

6 Numerical applications

The performance of these schemes with DR-BEM formulation is analyzed through the simulation of the dynamic response for two rods. The first one has a constant section and another one has a linearly variable section. Both rods are fixed at one of their ends and subjected to a unit constant longitudinal impact, permanently applied to the other end.

6.1 Constant section rod

The geometrical characteristics, as well as some discretization features, are shown in Fig. (1). Other meshes, not indicated in the figure, follow the same regularity pattern. In the numerical simulations, the boundary was discretized by 18, 36 and 72 boundary elements (BE). It was also analyzed the influence of the inclusion of poles (IP) in the accuracy of results.

Firstly, the results obtained through Wilson-\(\theta\) method are shown. This scheme generated similar response as those obtained through the Houlbolt’s method, but the parametric control of the dissipation allows the Wilson-\(\theta\) scheme to have a larger flexibility in the choice of the time step, especially in the most refined meshes. In Fig. (2) the displacement representation through Wilson-\(\theta\) using meshes of 18 and 72 BE can be observed. Good agreement can be observed between the numerical and analytic results, especially in the more discretized model.

Figure 1: Geometrical characteristics and some meshes used for discretization.

Figure 2: Displacements for Wilson-\(\theta\):18 and 72 BE, no IP, \(\Delta t=0.7s\) and \(\theta =2\).
Numerical representation of tractions offers greater difficulty and the boundary refinement is necessary to improve the response, as shown in Fig. (3). But the main factor to improve considerably the representation of tractions is the inclusion of internal points, that serve like additional interpolation points. Fig (4) presents results for tractions using meshes of 18 and 72 BE with 21 IP and Δt = 0.3 s. Another factor is the appropriate choice of the time step size (Δt). The simulations showed that due to the parametric control of fictitious damping, this method can be used with smaller time steps than those used in the Houbolt’s algorithm, especially for more refined meshes.

The second algorithm used is the CHL-β method, characterized by high rates of numerical dissipation. This algorithm also possesses dissipation parametric control. As in the previous case, the displacements representation is more accurate, but the high rates of dissipation quickly deteriorate the results, especially in the strip of β values proposed to the use with the FEM. Fig. (5) shows the influence of boundary discretization for the displacement using meshes of 18 and 72 BE and fig (6) shows the related tractions response.

Figure 3: Traction at Point B. Traction for Wilson-θ: 18 and 72 BE, no IP, Δt = 0.7 s and θ = 2.

Figure 4: Traction at Point B. Traction for Wilson-θ: 18 and 72 BE, Δt = 0.3 s, θ = 2 and 21 IP.

Figure 5: Displacements for CHL-β: 18 and 72 BE, no IP, Δt = 0.3s and β = 1.
To improve the solution it is necessary the appropriately choose the parameters that regulate the dissipation. Better results are achieved with the use of smaller time steps and values for $\beta$. The simulations show that it is possible to use a wide range of values for $\beta$. This range is very different to the one stipulated in the application of CHL-$\beta$ to FEM, as shown in Fig. (7).

The last algorithm used is the HHT-$\alpha$ method. This algorithm is characterized by admitting the use, without loss in the precision, of much larger steps than those allowed in the other algorithms used in this work. Fig. (8) presents results for the displacement showing the influence of boundary discretization and Fig. (9) displays the effect of inclusion of IP, considering 6 and 10 IP. The results do not show any accuracy improvement related to the previous schemes.
6.2 Linearly variable section rod

The last problem simulated is a linearly variable section rod. Meshes of 58, 117 and 294 BE were used with the inclusion of 38 IP. The geometrical characteristics and some meshes are presented in Fig. (10).

The difference between internal and external boundary dimensions makes this numerical simulation very difficult. In the Wilson-θ method, the value of θ capable to generate a stable response varies considerably in relation to the case of the constant section rod. For example, for a mesh of 58 BE with 38 IP and Δt = 0.2 s stable results were obtained only starting at θ = 7.5. Very good results were obtained for the displacements in these conditions, but reasonable response for tractions were achieved just for time step higher than 0.4 s, as shown in fig (11). However, the results present excessive rounding. In this figure it can be also noticed that the use of more refined meshes did not improve.

Figure 11: Traction for Wilson-θ: 58 and 117 BE, Δt=0.5 s and θ=7.5.
In CHL-β method, the range of values for β is a lot besides that established for FEM. For instance, for a mesh of 58BE with 38 IP and Δt = 0.2 s, the integration is possible only from β = 7.5. Fig (12) shows the displacement and the traction representations in these conditions. Good agreement is observed among the numerical and the analytical results for the displacement. But for tractions the behavior is different. In the same way of previous algorithms, excessive rounding is observed in the numerical solution.

The HHT-α method is very attractive due possibility of using large time steps, although in the traction representation it is not possible to obtain very significant improvements. For a mesh of 58 BE, 38 IP and α = -0.3 it is possible to make the integration from Δt = 4 s and in Fig (13) it is noticed that in these conditions the displacement representation is accurate, but the tractions have excessive rounding, although presenting some improvement in relation to the other methods.

**Figure 12:** Displacement and traction for CHL-β: 58 BE, 38 IP, Δt = 0.2s, β = 7.5.

**Figure 13:** Displacement and traction: 117 BE, 38 IP, Δt = 5 s and α = -0.3.

### 7 Conclusions

For all methods, a good level of accuracy was achieved for the displacements response without meaningful difficulties. But for traction calculations it was necessary to carefully choose the numerical parameters, especially those that regulate the algorithm dissipation.

In the Wilson-θ method, the appropriate rates of dissipation allows the inclusion of larger number of poles, improving the integration conditions in relation to the Houbolt’s schemes. The simulations showed that both schemes produce quite similar results, but the larger flexibility in the inclusion of
dissipation allows obtaining better results in the Wilson-θ method, especially with smaller time steps.

The CHL-β method presents the largest rate of reduction. Its results were reasonable only in the primordial cycle, because the amount of fictitious damping is intense and varies sensibly with small variations in the step size. It is possible to obtain results of considerable precision with the use of larger values of β combined with small values of Δt.

The last algorithm presented in this work, the HHT-α method, is interesting for the possibility of using larger steps to obtain results with similar precision to the other methods. In the tapered rod problem, this method produces the better traction representation, in spite of the solution continues presenting rounding.

References


