Initial conditions contributions in frequency domain analysis: a new BEM approach

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Abstract

This work presents a methodology to consider the contribution of initial velocity and displacement in the frequency domain formulation of the BEM. The paper describes how initial conditions can be replaced by pseudo-forces represented by generalized functions. The generation of such pseudo-forces and their discrete Fourier Transform (Direct DFT, and Inverse FT) are the subject of a short discussion and explanation. At the end of the paper two examples are presented in order to show the accuracy of the proposed approach.

1 Introduction

Several BEM formulations have been developed in order to solve transient problems, governed by the scalar wave equation. For general purposes, time-domain (TD) and transformed-domain formulations can be employed to solve scalar wave propagation problems. Time-domain BEM formulations (TD-BEM), e.g. Mansur [1], Dominguez [2], Antes & Estorff [3], Carrer & Mansur [4], provide good representation of the causality condition and lead to very accurate results.

Transformed-domain BEM formulations are also the subject of intense research work and are mainly related to the frequency-domain approach, e.g. Dominguez [2], Gaul & Wenzel [5]. When employing these formulations, the problem is initially solved in the transformed-domain and, if necessary, the solution in the time-domain can be obtained by means of a suitable inverse transformation procedure.

This work presents a BEM frequency-domain formulation that takes into account the initial conditions contribution in the solution of problems governed
by the scalar wave equation. Known frequency-domain formulations \cite{6,7,8,9}, do not take into account non-null initial conditions. Here, this difficulty was overcome in a methodology that consists of replacing the initial conditions (related to the potential and its time derivative) by pseudo-forces, represented by generalized functions. A detailed discussion concerning the various aspects of this methodology is carried out. Linear boundary elements and linear triangular cells have been employed in the BEM formulation; the reader is referred to Mansur \cite{1} and Carrer \& Mansur \cite{10} for general aspects concerning boundary and domain integration. At the end of the article, two examples are presented to validate the formulation and verify its accuracy and robustness.

2 Boundary element formulation in the frequency domain

The Helmholtz equation for an elastic and homogeneous medium \( \Omega \) can be written as, Morse \& Feshbach \cite{11}:

\[
\nabla^2 U(X) + \gamma^2 U(X) = 0
\]

In eqn. (1) \( X=(x,y) \) and \( \gamma \) is the wavenumber, computed by \( \gamma = \omega / c \), where \( \omega \) is the frequency and \( c \) is the wave propagation velocity. Here, only Newman and Dirichlet boundary condition are considered, i.e.,

\[
U(X) = \bar{U}(X) \quad X \in \Gamma_u \quad ; \quad P(X) = \frac{\partial U(X)}{\partial n} = \bar{P}(X) \quad X \in \Gamma_p
\]

where \( \Gamma \) is the boundary and \( \Gamma = \Gamma_u \cup \Gamma_p \). The starting BEM equation corresponding to eqn. (1) in \( \Omega \cup \Gamma \), for 2D problems, can be written as \cite{2}:

\[
c(\xi)U(\xi) = \int_{\Gamma} U(X) p^*_u(X,\xi) \, d\Gamma - \int_{\Gamma} P(X) u^*_u(X,\xi) \, d\Gamma
\]

Eqn. (3) is valid for boundary nodes \( \xi (\xi \in \Gamma) \) and for internal points, for the latter, \( c(\xi) = 1 \). In the above expression, \( u^*_u(X,\xi) \) is the fundamental solution and \( p^*_u(X,\xi) = \partial u^*_u(X,\xi) / \partial n \). Their expressions are given by, Morse \& Feshbach \cite{11}:

\[
u^*_u(r,\gamma) = \frac{-i}{4} H_0^{(1)} (\gamma r) ; \quad p^*_u(r,\gamma) = \frac{i \gamma}{4} H_1^{(1)} (\gamma r) \frac{\partial r}{\partial n}
\]

where \( r = |X-\xi| \) is the Euclidian distance between field and source points, \( X \) and \( \xi \) respectively. The application of the discretized version of eqn. (3) to all boundary nodes produces the following system of algebraic equations:

\[
Hu = Gp
\]

Taking into account the boundary conditions (2), the system of equations (5) can be written according to:

\[
AX = Y
\]

After solving the system of equations (6), all the boundary variables (potential and flux) become known and the potential at internal points can be computed from eqn. (3) if one takes \( c(\xi) = 1 \).

When the domain presents sources, says \( B(X) \), the BEM integral equation becomes:
The computation of the boundary unknowns follows a similar pattern to that indicated by eqn. (6); the only difference now consists of the presence of a known vector $B$ that contains the source contribution. The corresponding system of equations is given below:

$$AX_b = Y + B$$

(8)

Now, one is able to solve $U(X)$ from eqn. (1), for a given wavenumber $\gamma$ with the boundary conditions given by (2), zero initial conditions and the presence or not of domain sources $B(X)$.  

3 Transient analysis in the frequency domain

Frequency-domain analysis using standard Discrete Fourier Transform (DFT) or Fast Fourier Transform (FFT) algorithms [6, 7, 8, 9] can only be carried out if damping exists, due to the singularities in the frequency spectrum at the natural frequencies, and also because undamped systems response does not follow the theoretical decay condition as $t \to \infty$, which renders Fourier transform possible. Thus it is necessary to consider damping. According to Morse & Feshbach [11], in the presence of damping, the wave equation can be written as:

$$\nabla^2 u(X,t) - c_v \frac{\partial u(X,t)}{\partial t} - \frac{1}{c^2} \frac{\partial^2 u(X,t)}{\partial t^2} = b(X,t)$$

(9)

where $c$ is the wave propagation velocity, and $c_v$ is the constant viscous damping. Boundary conditions are given by:

$$u(X,t) = \bar{u}(X,t) \mbox{ } X \in \Gamma_u \quad ; \quad p(X,t) = \frac{\partial u(X,t)}{\partial n} = \bar{p}(X,t) \mbox{ } X \in \Gamma_p$$

(10)

Supplementary, for the initial conditions one has in $\Omega \cup \Gamma$:

$$u(X,0) = u_0(X) \quad ; \quad v(X,0) = \frac{\partial u(X,t)}{\partial t} \Big|_{t=0} = v_0(X)$$

(11)

Assuming that the function $b(X,t)$ in eqn. (9) is a combination of simple-harmonic components, it can be decomposed into its frequency components by means of the Fourier transform [6, 7, 8, 9]. Applying the Fourier transform to eqn. (9), a familiar equation for $U(X)$ with the same form of eqn. (1), called generalized Helmholtz equation, is obtained and the wavenumber $\gamma$ now is a complex variable:

$$\frac{\partial^2 U(X)}{\partial X^2} + \gamma^2 U(X) = B(X) \quad ; \quad \gamma = \left(\frac{\omega^2}{c^2} - i\omega c_v\right)^{\frac{1}{2}}$$

(12)

The Fourier transforms of $u(X,t)$ and $b(X,t)$ are represented, respectively, by complex quantities $U(X) = U(X,\omega)$ and $B(X) = B(X,\omega)$. After applying the Fourier transform to the boundary conditions given by eqn. (10), expressions similar to those given by eqn. (2) are obtained. The BEM formulation follows the same
steps already shown until eqn. (7): one must note that the fundamental solution, in this case, is a function of the complex wavenumber $\gamma_c$.

3.1 Numerical procedure

For numerical purposes, the boundary $\Gamma$ is approximated by linear elements whereas the part of the domain $\Omega$, where non homogeneous initial conditions appear, say $\Omega_\alpha$, is approximated by linear triangular cells. If discrete values of the function $b(X,t)$ are known for $NT$ terms (time sampling), the discrete spectrum $B(X,\omega_m)$ can be obtained by means of the Discrete Fourier Transform (DFT) in $\omega_m$ frequencies equally spaced ($nt = 1, 2, ..., NT$). The time interval is calculated as $\Delta t = T/(NT)$, where $T$ is the period of $b(X,t)$. Until now, a problem depending on every frequency $\omega_m$ and governed by the generalized Helmholtz equation (12) was found. As a matter of fact, $NT$ boundary problems, analogous to that from section 2, are presently being solved: matrices $H$ and $G$ are generated $NT$ times and, for the computation of the unknowns, $NT$ systems like that represented by eqn. (8) are written. The responses at internal points, for every frequency $\omega_m$, are computed by following the standard BEM procedure, e.g., by employing eqn. (7). The responses at boundary nodes and internal points can be stored in matrix form and the final time-dependent responses ($u_b$ for instance) are computed by applying the Inverse Discrete Fourier Transform (IDFT).

4 Initial conditions contribution

Due to the linearity of the problem, the initial conditions contribution can be studied separately and the corresponding solutions added to the final response. In this section the 2D case will be discussed. The methodology developed by the authors is described below.

4.1 Initial displacement

The response corresponding to an initial displacement field $u_0(X)$ is computed by adding to the initial displacement field itself the response of the mechanical system to a force $-f_{u_0}(X)H(t - 0)$, where $H(t - 0)$ is the Heaviside function and $f_{u_0}(X)$ is a reactive static domain force obtained from eqn. (7) ($\omega = 0$), replacing $B(X)$ by $f_{u_0}(X)$ and considering that $u(X) = u_0(X)$ and $p(X) = p_0(X)$ are known.

The assemblage of the matrices $H$ and $G$, in this case, is made for $\gamma = 0$, and the corresponding expressions for $u^*(X,\xi)$ and $p^*(X,\xi)$ are given by [11]:

$$u^*(r) = \frac{-1}{2\pi} \ln(r) \quad ; \quad p^*(r) = -\frac{1}{\gamma} \frac{\partial r}{\partial n}$$

Due to the implicit periodicity of the Fourier transform approach, the time response produces by the force $f_{u_0}H(t - 0)$ can not be computed correctly, i.e., DFT algorithms represent periodic functions and can not calculate the response
due to a force applied at initial time and kept constant until the end of the analysis. This drawback can be overcome by considering an extended period equal to $2T_p$, and by assuming $f_{uo}$ constant within the time interval $[0, T_p]$ and null within the interval $[T_p, 2T_p]$.

Once the dependence on time of $-f_{uo}$ is established, as explained above, the Fourier transform of $-f_{uo}$ can be computed for $NT$ terms of time. This Fourier transform is denoted by $-F_{uo}$ and, for each frequency, a system similar to eqn. (8) is written as follows:

$$AX_{uo} = -F_{uo}$$

The response in the time domain, $x_{uo}$, is computed by applying IDFT to the matrix $M$ that stores all the frequency responses at boundary nodes and internal points. The final response $u_{id}$ (id means initial displacement) due to initial displacement $u_o$ is calculated by adding to the initial displacement the response related to $-f_{uo}$, i.e.:

$$u_{id} = x_{uo} + u_o$$

4.2 Initial velocity

The response due to the initial velocity $v_o(X)$ can be computed from the impulse force $c^2\Delta t \delta(t - 0)$, where $\Delta t$ is the time interval. The force $f_{vo}$ should be applied at each nodal point.

The Fourier transform $F_{vo}$ obtained from $f_{vo}$ vector in $NT$ terms is computed, and a procedure similar to that explained previously for $-F_{uo}$ is followed for $F_{vo}$; it means that the system below is found:

$$AX_{vo} = F_{vo}$$

The responses on frequency domain at the internal points are computed and the $NT$ time dependent responses $x_{vo}$ are determinate using the IDFT. Finally, the corresponding final response in the time domain with initial displacement and velocity contributions, and the presence of sources such as $b(X,t)$, is given by:

$$u = u_{id} + x_{vo} + u_b$$

5 Examples

Two numerical applications are presented next in order to verify the accuracy of the numerical results provided by the formulation presented in this work. The dimensionless parameter $\beta = c(\Delta t)l$ that gives a measure of the time-step length to be adopted to perform the analysis will be referenced, where $l$ is the smaller boundary element length. The Fast Fourier Transform (FFT) and the Inverse Fast Fourier Transform (IFFT) algorithms [6, 7, 8, 9] were used with the aim of reducing the computational effort. The bi-dimensional responses were compared with the corresponding one-dimensional responses, determined in the same manner using this new methodology.
The one-dimensional case consists of a rod of length \( L = 12 \). The Fourier (FFT) algorithm considered an extended period \( 2T_p = 1842.3808 \), and the one-dimensional responses are presented with \( NT = 8192 \) terms of the Fourier transform with a time interval \( \Delta t = 0.2249 \).

5.1 One-dimensional rod under displacement sinusoidal initial condition

The first example shown in Figure 1a consists of an one-dimensional rod under prescribed initial displacement with sinusoidal space dependence, where \( A \) is the amplitude. Figure 1b shows the boundary mesh used for this case. The selected internal points at which the numerical responses will be computed are on the horizontal line \( y = a/4 \). Twenty four linear elements, resulting in \( NN = 28 \) nodes, were employed in the boundary discretization and 64 linear cells were used for the domain discretization. The data for the analysis were \( c = 1 \), the dimension \( a = 12 \) and \( \zeta = 0.016 \).

In Figure 2 the responses due to displacement sinusoidal initial condition are compared with the 1D responses. The one-dimensional domain was divided into eight linear elements and \( NN = 9 \) nodes for this effect, this one-dimensional case consisted of a string of length \( L = 12 \) fixed at its extremities, i.e. null displacement prescribed at \( x = 0 \) and \( x = L \). In this example for the 2D case \( NT = 2048 \) Fourier coefficients were used and the same extended period that the 1D case with a time interval \( \Delta t = 0.8996 \).

5.2 Square membrane under prescribed initial velocity

This example consists of the square membrane shown in Figure 3a under prescribed initial velocity. Figure 3b shows the boundary mesh used with \( NN = 64 \) nodes and the internal point \( C \) selected. For this analysis 8 linear triangular cells were used and \( c = 1 \).
Figure 2: Comparison of displacements for 1D and 2D cases under displacement sinusoidal initial condition at different times.

Figure 3: Square membrane analysis. (a) Geometry and boundary conditions. (b) boundary and domain discretization.

In Figure 4 the response at point C is compared with the dampingless analytical response for $\zeta = 0.025$, $\Delta t = 0.025$ and $NT = 8192$, the extended period is $2Tp = 205$ and $\beta = 0.4$. 
Figure 4: BEM and analytical displacement responses for the membrane under prescribed initial condition, for $\zeta = 0.025$.

6 Conclusions

In the present work, an original formulation was developed with the aim of solving transient problems with non-homogeneous initial conditions in the frequency domain. The methodology proved to be very efficient, as demonstrated by the numerical results presented, and consists, basically, of finding pseudo-forces equivalent to known initial conditions. The basic steps were presented in details in the corresponding sections.

It is the authors' opinion that the methodology presented here can become a powerful tool for solving problems in 2D and 3D with BEM, FEM, FDM and other numerical methods.

References


