Abstract

This paper presents fundamental solutions for the response of homogeneous three-dimensional layered solid formations subjected to spatially sinusoidal harmonic heat line sources. The formulation is performed in the frequency domain, while time solutions are computed using inverse Fourier transforms. Interference from aliasing is prevented by the use of complex frequencies. The formulas are not only interesting per se, but they are also useful as benchmark solutions for numerical applications. When the solutions are applied in conjunction with numerical methods, such as the Boundary Element Method, discretization of the material interfaces is not necessary. They may well prove to be very useful in many engineering applications, such as the calculation of the heat insulation provided by solid layered walls and slabs.

1 Introduction

Carslaw and Jaeger's book [1] is the most important reference for heat transfer as it contains a set of analytical solutions and Green's functions for the diffusion equation. It also contains a wide-ranging review of numerical methods, which can be grouped according to how the time-dependent terms are handled. The "time marching" approach is the first one in which the solution is assessed step by step at successive time intervals after an initially specified state has been described. The second applies the Laplace transform of the time in which the diffusion equation becomes an elliptical one. A numerical transform inversion is employed to calculate the physical variables in the real space after the solution has been obtained for a succession of values of the transform parameter. The Laplace transform technique has been extensively used for solving diffusion
problems, but small truncation errors can be magnified in the inversion process and so the accuracy depends on an efficient and precise inverse transform. Several inversion methods have been proposed over the years, such as the Stehfest algorithm [2].

This work presents Green's functions for calculating the heat radiated by a spatially sinusoidal, harmonic heat line source placed in solid layered media. These fundamental solutions, relate the heat field variables (fluxes or temperatures) at some location in the solid domain caused by a heat source placed elsewhere in the media.

The approach used requires the Green's function for the unbounded media to be written as a superposition of heat plane waves, following a technique similar to the one used first by Lamb [3] for the propagation of elastodynamic waves in a two-dimensional media. Other authors, such as Bouchon [4] and Tadeu and António [5], used an analogous approach to compute three-dimensional elastodynamic fields using a discrete wave number representation.

The Green's functions for a solid layered formation are written as the sum of the heat source terms equal to those in the full-space and the surface terms needed to satisfy the boundary conditions at the interfaces.

The final heat field is obtained by adding the heat source terms, equal to those in the full-space, to a set of surface terms, originated within each solid layer and at each interface. The amplitudes of these surface terms are known after assigning the required boundary conditions, i.e. continuity of temperatures and normal fluxes between solid layers, and null normal fluxes or null temperatures at the outer surface.

The expressions presented in this work make it possible to calculate the heat field inside a layered solid medium, without having to fully discretize the interior domain, which is necessary when using numerical techniques, such as the finite differences method, or even needing to discretize the free surface using boundary elements techniques.

2 3D problem formulation and Green's functions in an unbounded medium

The solution of transient heat conduction in solids is expressed by the diffusion equation

$$\nabla^2 T = \frac{1}{K} \frac{\partial T}{\partial t}$$

where

$$\nabla^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

is the thermal diffusivity, $k$ is the thermal conductivity, $\rho$ is the density and $c$ is the specific heat. Applying a Fourier transform in the time domain, one obtains...
where \( i = \sqrt{-1} \) and \( \omega \) is the frequency. Equation 2 is a Helmholtz equation comparable to the one applied when solving acoustic problems, where \( \omega / (\text{velocity of pressure waves}) \) corresponds to \( \sqrt{-\frac{i\omega}{K}} \) in the diffusion equation. Thus, the transient heat propagation solution can be perceived as a propagation of heat waves.

The fundamental solution of equation 2 for a heat point source in an unbounded medium, located at \((x_0, y_0, 0)\), \( p(\omega, x, y, z, t) = \delta(x - x_0) \delta(y - y_0) \delta(z) e^{i\omega t} \) where \( \delta(x - x_0) \), \( \delta(y - y_0) \) and \( \delta(z) \) are Dirac-delta functions, can be expressed as

\[
\hat{T}_f(\omega, x, y, z) = \frac{1}{2k} \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}} e^{-\frac{i\omega}{K} \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2}}
\]

Owing to the computational cost involved in solving many 3D problems, when the geometry of the problem remains constant along one direction \( z \) it is easier to express the full 3D problem as a summation of simpler 2D solutions. This is achieved by applying a Fourier transformation along that direction, expressing this as a summation of 2D solutions with different spatial wavenumbers \( k_z \) (Tadeu and Kausel [6]). The application of a spatial Fourier transformation along the \( z \) direction in equation 2 leads to the following equation

\[
\left( \tilde{\nabla}^2 + \left( \frac{-i\omega}{K} - (k_z)^2 \right) \right) \tilde{T}(\omega, x, y, k_z) = 0
\]

with \( \tilde{\nabla}^2 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \).

The fundamental solution for the fundamental equation for a heat point source (equation 3) is found by applying a spatial Fourier transform to it, in the \( z \) direction,

\[
\tilde{T}_f(\omega, x, y, k_z) = \frac{-i}{4k} H_n \left( \frac{-i\omega}{K} \sqrt{(k_z)^2} r_0 \right)
\]

where \( H_n(\cdot) \) are Hankel functions of the second kind and order \( n \), and \( r_0 = \sqrt{(x-x_0)^2 + (y-y_0)^2} \). This can be seen as the response to a spatially varying heat line source of the form \( p(\omega, x, y, k_z, t) = \delta(x - x_0) \delta(y - y_0) e^{i(\omega t - k_z z)} \).

The full three-dimensional solution is then achieved by applying an inverse Fourier transform along the \( k_z \) domain. This inverse Fourier transformation can be expressed as a discrete summation if we assume the existence of virtual
sources, equally spaced at $L_z$, along $z$, which enables the solution to be obtained by solving a limited number of two-dimensional problems.

$$
\mathbf{T}(\omega, x, y, z) = \frac{2\pi}{L} \sum_{m=-M}^{M} \mathbf{T}(\omega, x, y, k_{zm}) e^{-ik_{zm}z}
$$

with $k_{zm}$ being the axial wavenumber given by $k_{zm} = \frac{2\pi}{L_z}m$. The distance $L_z$ must be large enough to prevent spatial contamination from the virtual sources (Bouchon and Aki [7]). The authors adopted a similar method to analyze the wave propagation inside seismic prospecting boreholes (Tadeu et al. [8]) and the outdoor propagation of sound waves in the presence of obstacles (Godinho et al. [9]).

These same equations can be written as a continuous superposition of heat plane waves. Equation (5), which results when a spatially sinusoidal harmonic heat line source is applied at the point $(x_0, y_0)$ along the $z$ direction, is then given by the expressions,

$$
\tilde{T}_f(\omega, x, y, k_z) = \frac{-i}{4\pi k} \int_{-\infty}^{\infty} \left( \frac{e^{-iv_0|y-y_0|}}{v} \right) e^{-ik_z(x-x_0)} dk_x
$$

where $v = \sqrt{\frac{-i\omega}{K} - k_z^2 - k_x^2}$ with $(\text{Im}(v) \leq 0)$, and the integration is performed with respect to the horizontal wave number ($k_x$) along the $x$ direction.

The integral in the above equation can be transformed into a summation if an infinite number of such sources are distributed along the $x$ direction, at equal intervals $L_x$. The above equation can then be written as

$$
\tilde{T}_f(\omega, x, y, k_z) = E_0 \sum_{n=-\infty}^{\infty} \left( \frac{E}{v_n} \right) E_d
$$

where $E_0 = \frac{-i}{2kL_x}$, $E = e^{-i\sqrt{v_0|y-y_0|}}$, $E_d = e^{-ik_{xn}(x-x_0)}$, $v_n = \sqrt{\frac{-i\omega}{K} - k_z^2 - k_{xn}^2}$ with $(\text{Im}(v_n) \leq 0)$, $k_{xn} = \frac{2\pi}{L_x}n$, which can in turn be approximated by a finite sum of equations ($N$). Notice that $k_z = 0$ corresponds to the two-dimensional case.

The heat in the spatial-temporal domain is calculated by applying a numerical inverse fast Fourier transform in $k_z$, in the frequency domain. The computations are performed using complex frequencies with a small imaginary part of the form $\omega = \omega - i\eta$ (with $\eta = 0.7\Delta\omega$, and $\Delta\omega$ being the frequency step) to prevent interference from aliasing phenomena. In the time domain, this effect is removed by rescaling the response with an exponential window of the form $e^{\eta t}$. The temporal variation of the source can be arbitrary.
2.1 Verification of the solution

The procedures presented in the previous section were implemented and verified by using them to calculate the one, two and three-dimensional exact time solutions for a unit heat source placed in an unbounded medium. The exact solution of the diffusion equation (1) in an unbounded medium in the time domain, describing the temperature field generated by a unit heat source applied at point \((x_0, y_0, z_0)\) at time \(t = t_0\), is:

\[
T(t, x, y, z) = \frac{1}{\rho c(4\pi k\tau)^{d/2}} e^{\left(-\frac{x^2}{4k\tau}\right)} i f \ t > t_0
\]  

(9)

where \(\tau = t - t_0\), \(r_{00}\) is the distance between the source point and the field point \((x, y, z)\), and \(d = 3\), \(d = 2\) and \(d = 1\) when in the presence of a three, two and one-dimensional problem, respectively [1].

A homogeneous unbounded solid medium with thermal material properties where \(k = 1.4 \text{ W} \cdot \text{m}^{-1} \cdot \text{C}^{-1}\), \(c = 880.0 \text{ J} \cdot \text{Kg}^{-1} \cdot \text{C}^{-1}\) and \(\rho = 2300 \text{ Kg} \cdot \text{m}^{-3}\), is excited by a unit heat source applied at \((x = 0.0 \text{ m}, y = 0.0 \text{ m}, z = 0.0 \text{ m})\) when \(t = 277.8 \text{ h}\). Figure 1 shows the temperature calculated, using equation 9, along a line of 40 receivers placed from \(y = -1.5 \text{ m}\) to \(y = 1.5 \text{ m}\), for a plane \((d = 1)\), cylindrical \((d = 2)\) and spherical \((d = 3)\) unit heat source, at different times.

The Green's functions proposed in this paper are used to compute the heat responses at these same receivers. Computations are carried out in the frequency range \([0, 1024 \times 10^{-7} \text{ Hz}]\) with a frequency increment of \(\Delta \omega = 10^{-7} \text{ Hz}\), which defines a time analysis of \(T = 2777.8 \text{ h}\). The solution for a plane unit heat source propagating along the \(y\) axis has been modeled ascribing \(k_z = 0\) and \(k_{xn} = 0\) to equation (8), multiplied by \(L_z\). For a cylindrical unit heat source the solution has been computed using equation (5), ascribing \(k_z = 0\), while the response for a spherical unit heat source has been obtained using equation (3) divided by \(2\pi\). Complex frequencies of the form \(\omega_c = \omega - 0.7i\Delta\omega\) have been used to avoid the aliasing phenomenon. In Figure 1, the solid line represents the solution given by equation (9) while the marks show the response calculated using the proposed Green's functions. The agreement between these two solutions is excellent.

The cylindrical and the spherical unit heat source responses have also been obtained by calculating discrete summations over wavenumbers, following equations (6) and (8), which is mathematically equivalent to adding sources at spatial intervals \(L_x\), \(L_z\). The spatial period has been set as \(L_x = L_z = 2\sqrt{k/\rho c \Delta f}\). The agreement between solutions has also proved to be excellent.
3 Green’s functions in a layered formation

The Green’s functions for a multi-solid layer bounded by two semi infinite solid media are found using the necessary boundary conditions at the solid-solid interfaces.

The system to be analyzed is a medium built from a set of \( m \) solid flat layers of infinite extent bounded by flat, semi-infinite, solid media (top: semi-infinite medium (medium 0) and bottom, semi-infinite medium (medium \( m+1 \))). The layers can possess different thermal material properties and be of different thicknesses. This system is excited by a spatially sinusoidal heat source placed in the first layer (medium 1). The response is achieved by adding the direct contribution of the heat source and the surface heat terms generated at all interfaces.

For the solid layer \( j \), the heat surface terms on the upper and lower interfaces can be expressed as

\[
\tilde{T}_{ji} (\omega, x, y, k_z) = E_{0j} \sum_{n=\infty}^{n_{\text{max}}} \left( \frac{E_{ij}}{V_{nj}} A_{nj} \right) E_d
\]
\[
\tilde{T}_{j2}(\omega, x, y, k_z) = E_{0j} \sum_{n=-\infty}^{n=\infty} \left( \frac{E_{j2}}{v_{nj}} A_n^b \right) E_d
\]

\[
E_{0j} = \frac{-i}{2k_jL_j} \quad E_{j1} = e^{-iv_{nj}y-\sum_{i=1}^{n-1} h_i} \quad E_{j2} = e^{-iv_{nj}y-\sum_{i=1}^{n} h_i}
\]

where \(v_{nj} = \sqrt{\frac{-i\omega}{K_j}} - k_z^2 - k_{on}^2\), with \(\text{Im}(v_{nj}) \leq 0\) and \(h_i\) is the thickness of the layer \(l\).

Meanwhile, \(K_j = \frac{k_j}{\rho_j c_j}\) is the thermal diffusivity in the solid medium \(j\) (\(k_j\), \(\rho_j\) and \(c_j\) represent the thermal conductivity, the density and the specific heat of the material in the solid medium \(j\), respectively). The heat surface terms produced at interfaces \(1\) and \(m+1\), governing the heat that propagates through the top and bottom semi-infinite media, are respectively expressed by

\[
\tilde{T}_{02}(\omega, x, y, k_z) = E_{00} \sum_{n=-\infty}^{n=\infty} \left( \frac{E_{01}}{v_{n0}} A_n^b \right) E_d
\]

\[
\tilde{T}_{(m+1)2}(\omega, x, y, k_z) = E_{1(m+1)} \sum_{n=-\infty}^{n=\infty} \left( \frac{E_{(m+1)1}}{v_{n(m+1)}} A_n^t \right) E_d
\]

(11)

The final system matrix established takes into account the coupling between the different layers, so that the heat created simultaneously by the source and surface terms leads to the continuity of fluxes and temperatures along the \(m+1\) solid interfaces. Thus a system of \(2(m+1)\) equations in the \(2(m+1)\) unknown coefficients is defined \((Fa = b)\) for each value of \(n\),

\[
\begin{bmatrix}
-1 & -1 & e^{-i\nu_{n1}} & \ldots & 0 & 0 & 0 \\
-\frac{k_jv_{n0}}{k_jv_{n1}} & -1 & 0 & \ldots & 0 & 0 & 0 \\
0 & e^{-i\nu_{n1}} & -1 & \ldots & 0 & 0 & 0 \\
0 & \frac{1}{k_jv_{n1}} & 0 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & -1 & e^{-i\nu_{m1}} & 0 \\
0 & 0 & 0 & \ldots & -\frac{k_jv_{nm}}{k_jv_{n1}} & -e^{i\nu_{m1}} & 0 \\
0 & 0 & 0 & \ldots & e^{-i\nu_{m1}} & e^{i\nu_{m1}} & 0 \\
0 & 0 & 0 & \ldots & \frac{k_jv_{nm}}{k_jv_{nm}} & -1 & 0 \\
0 & 0 & 0 & \ldots & e^{-i\nu_{m1}} & 1 & 1 \\
0 & 0 & 0 & \ldots & \frac{k_jv_{nm}}{k_jv_{nm}} & k_jv_{nm} & k_{m+1}v_{n(m+1)} \\
\end{bmatrix}
\begin{bmatrix}
A_n^b \\
A_{n1}^t \\
A_{n1}^b \\
A_{n1}^t \\
\ldots \\
A_{nm}^b \\
A_{nm}^t \\
A_{nm}^b \\
A_{nm}^t \\
\end{bmatrix}
= \begin{bmatrix}
-\frac{e^{-i\nu_{m1}y_0}}{k_jv_{n1}} \\
\frac{e^{-i\nu_{m1}y_0}}{k_jv_{n1}} \\
\ldots \\
\frac{e^{-i\nu_{m1}y_0}}{k_jv_{n1}} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

(12)

Solving the system gives the amplitude of the surface terms in each solid interface. The Green's functions for a solid layer formation are expressed as the sum of the source terms and these surface terms, leading to the following expressions,
top semi-infinite medium (medium 0)

\[ \tilde{T}(\omega, x, y, k_z) = E_{00} \sum_{n=-\infty}^{n=+\infty} \left( \frac{E_{01}}{V_{n0}} A_n^{b0} \right) E_d \quad \text{if } y < 0 \]

solid layer 1 (source position)

\[ \tilde{T}(\omega, x, y, k_z) = -\frac{i}{4k_1} H_0(K_{11} r_0) + E_{01} \sum_{n=-\infty}^{n=+\infty} \left( \frac{E_{11}}{V_{n1}} A_n^{i1} + \frac{E_{12}}{V_{n1}} A_n^{b1} \right) E_d \quad \text{if } 0 < y < h_1 \]

solid layer j (j ≠ 1)

\[ \tilde{T}(\omega, x, y, k_z) = E_{0j} \sum_{n=-\infty}^{n=+\infty} \left( \frac{E_{j1}}{V_{nj}} A_n^{i1} + \frac{E_{j2}}{V_{nj}} A_n^{b1} \right) E_d \quad \text{if } \sum_{i=1}^{i=j} h_i < y < \sum_{i=1}^{i=j} h_i \]

bottom semi-infinite medium (medium m + 1)

\[ \tilde{T}_{(m+1)2}(\omega, x, y, k_z) = E_{0(m+1)} \sum_{n=-\infty}^{n=+\infty} \left( \frac{E_{(m+1)2}}{V_{n(m+1)}} A_n^{i(m+1)} \right) E_d \quad (13) \]

The solution for a heat source located in a different solid layer can be calculated, maintaining the same matrix system (F) and changing only the independent terms corresponding to the direct incident field (B). Thus, the derivation of the full system of equations is basically straightforward, and for this reason is not presented here.

3.1 Verification of the solution

The results obtained with the analytical expressions presented above are compared with those calculated using the BEM model, which involves the discretization of all solid interfaces, making use of the Green’s functions for a full space. The BEM code has in turn been validated for the case of circular ring inclusions, for which analytical solutions have already been derived (not included here).

The unlimited discretization of the solid interfaces in the BEM model was avoided by introducing a damping factor, using complex frequencies with a small imaginary part of the form \( \omega_c = \omega - i\eta \) (with \( \eta = 0.7\Delta\omega \)) [Bouchon and Aki [10], Phinney [11]]. The contribution of boundary elements to the response is significant up to a defined distance and in the presence of a certain value of damping, but are otherwise unnecessary. In the present case the elements are distributed along the surface up to \( L_{\text{dis}} = 2\sqrt{k_j/\left(\rho_j c_j \Delta f\right)} \), using the thermal material properties from the solid medium that lead to the largest spatial distance.

The results for a solid flat layer bounded by two semi-infinite solid media are displayed in Fig. 2. The thermal material properties used are presented in Table 1.
A harmonic point source applied in the solid layer medium at point \((x = 0.0 \text{ m}, y = 1.2 \text{ m})\) heats the solid structure. The computations are accomplished in the frequency domain from \(0 \text{ Hz}\) to \(128 \times 10^{-7} \text{ Hz}\), with a frequency step of \(\Delta \omega = 1 \times 10^{-7} \text{ Hz}\). The imaginary part of the frequency has been set to \(\eta = 0.7\Delta \omega\). The results have been checked by calculating the response for a single value of \(k_z (k_z = 0.4 \text{ rad/m})\). The real and imaginary parts of the responses at receiver 1 \((x = 0.1 \text{ m}, y = 0.75 \text{ m})\) and receiver 2 \((x = 0.1 \text{ m}, y = 5.75 \text{ m})\) are displayed in Fig. 2. The solid lines represent the analytical responses, while the marked points correspond to the BEM solution. The square and the round marks designate the real and imaginary parts of the responses, respectively.

### Table 1. Thermal material properties.

<table>
<thead>
<tr>
<th></th>
<th>Solid layer (concrete)</th>
<th>Lower solid medium (steel)</th>
<th>Top solid medium (steel)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Thermal conductivity</strong></td>
<td>(k_1 = 1.4 \text{ W.m}^{-1} \cdot \text{C}^{-1})</td>
<td>(k_2 = 63.9 \text{ W.m}^{-1} \cdot \text{C}^{-1})</td>
<td>(k_0 = 63.9 \text{ W.m}^{-1} \cdot \text{C}^{-1})</td>
</tr>
<tr>
<td><strong>Density</strong></td>
<td>(\rho_1 = 2300 \text{ Kg.m}^{-3})</td>
<td>(\rho_2 = 7832 \text{ Kg.m}^{-3})</td>
<td>(\rho_0 = 7832 \text{ Kg.m}^{-3})</td>
</tr>
<tr>
<td><strong>Specific heat</strong></td>
<td>(c_1 = 880 \text{ J.Kg}^{-1} \cdot \text{C}^{-1})</td>
<td>(c_2 = 434 \text{ J.Kg}^{-1} \cdot \text{C}^{-1})</td>
<td>(c_0 = 434 \text{ J.Kg}^{-1} \cdot \text{C}^{-1})</td>
</tr>
</tbody>
</table>

The two solutions are seen to be in very close agreement, and equally good results were obtained from tests in which sources and receivers were placed at different locations.

![Figure 2: One solid layer bounded by two semi-infinite solid media: a) Receiver 1. b) Receiver 2.](image)

### 4 Conclusions

The analytical solutions developed for calculating the heat propagation in unbounded, and layered media, in presence of a spatially sinusoidal harmonic heat line source, seem to be well suited to transient heat conduction analyses. In
addition to the frequency responses, time signatures were calculated applying inverse Fourier transformations, using complex frequencies in order to avoid the aliasing phenomenon.

The solutions presented here are found to be in very close agreement with other analytical solutions, in the case of unbounded media, and with the Boundary Elements solution in the case of layered media.

The analytical solutions presented in this paper are not only intrinsically interesting, but if employed in conjunction with numerical methods, such as the BEM, they may also prove to be very useful in many engineering applications, such as the calculation of the thermal insulation provided by solid walls and slabs.

References


