

The detection of subsurface inclusions using internal measurements and genetic algorithms

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Abstract

An inverse problem is considered to identify the geometry of discontinuities in a conductive material with anisotropic conductivity from Cauchy data measurements taken on the boundary. In this study we propose a real coded genetic algorithm in conjunction with a boundary element method to detect an anisotropic inclusion, such as a circle, by a single boundary measurement. Numerical results are presented for both isotropic and anisotropic inclusions. The genetic algorithms based method developed in this paper is found to be a robust, efficient method for detecting the size and location of subsurface inclusions.

1 Introduction

We consider the inverse conductivity problem which requires the determination of an object D contained in a domain Ω from temperature measurements on the boundary $\partial\Omega$ and in the interior of the domain Ω . In inverse geometric problems, one requires to obtain information on defects, for example position and shape, or conductivity, from measurements of the temperature, T , and heat flux, $\frac{\partial T}{\partial n}$, on the material surface or in the interior of the domain. As an example, this problem models the determination of the shape, size and location of the anisotropic inner core of the Earth from measurements taken at its mantle. There are also other applications in electrical impedance tomography (EIT) or in nondestructive testing of materials using infrared scanning.

In the electrical impedance tomography (EIT) problem, which arises frequently in patient monitoring in hospitals, the shape, size and location of the object D are known, but its anisotropic conductivity K has to be identified. Alternatively, one

can assume that the conductivity K is known and only the shape, size and location of the object D have to be identified. Clearly, this latter inverse problem depends on the finiteness of Ω , i.e. bounded or unbounded, and the shape of D , e.g. convex, star-shaped, simply connected, polygonal, circular, cylindrical or spherical, for which one or two boundary measurements of the Dirichlet to Neumann map for the Laplace equation are sufficient to retrieve the domain D . If both D and K are unknown, the problem becomes more difficult. We investigate both the case when the conductivity K is known and the case when it is not known. The method employed is based on minimising an objective function which measures the fitness to the measured boundary and internal data given by a possible solution for the inclusion D .

We note that real coded genetic algorithms (RCGAs) have been successfully applied to the problem of the detection of inclusions in Mera *et al.* [1] using heat flux measurements. It is the purpose of this paper to extend this GA technique for the detection of isotropic inclusions using internal temperature measurements rather than boundary heat flux measurements. A complete description of genetic algorithm techniques has been provided by several authors, see for example Michalewicz [2]. A real coded genetic algorithm, similar to the one described in Michalewicz [2], is employed in this paper since it was found that algorithms of this type accurately model the continuous search space of numerical optimisation problems, have an increased rate of convergence with respect to the number of generations performed for multi-dimensional high precision numerical optimisation problems and enable fine local tuning for the unknown variables.

2 Mathematical formulation

Let Ω be a bounded domain of R^2 , with Lipschitz boundary, and D a subdomain compactly contained in Ω . The isotropic conductivity tensor $K = kI$ of the domain D is non-dimensionalised with respect to the isotropic constant conductivity tensor of the domain $\Omega - D$, so the refraction (transmission, conjugate) problem for the electrical potential T is given by

$$\nabla \cdot ((I + (K - I)\chi_D)\nabla T) = 0, \quad \text{in } \Omega \quad (1)$$

$$T = h, \quad \text{on } \partial\Omega \quad (2)$$

subject to refraction conditions related to the continuity of the temperature T and its heat flux densities $(\partial T / \partial n^-)$ and $(K \nabla T) \cdot \underline{n}^+$ across the interface ∂D , where $\underline{n}$, $\underline{n}^-$ and $\underline{n}^+$ are the outward unit normals to the boundaries $\partial\Omega$, $\partial(\Omega - D) - \partial\Omega$ and ∂D , respectively. It is well-known that the direct problem of finding $T \in H^1(\Omega)$ satisfying (1) and (2), when K and D are known, has a unique solution, see Ladyzhenskaya [3].

Assuming that K is known, the inverse conductivity problem requires the determination of D from some extra measurements. These additional measurements

may prescribe the heat flux on the boundary, i.e. the following additional boundary condition is imposed

$$\frac{\partial T}{\partial n}(\underline{x}) = q(\underline{x}) \quad \text{for } \underline{x} \in \partial\Omega \quad (3)$$

or they may specify the temperature at some interior points, for example along a curve inside the solution domain. In this case the additional measurements are given by

$$\frac{\partial T}{\partial n}(\underline{x}) = g(\underline{x}) \quad \text{for } \underline{x} \in \Gamma_1 \quad (4)$$

where $\Gamma_1 \subset \Omega$.

The inclusion detection problem can be reformulated as an optimisation problem if, for a given possible solution D for the inclusion, the direct problem (1)-(2) is solved to evaluate the heat flux on the outer boundary $T'_{calc} = \frac{\partial T}{\partial n}|_{\Gamma}$ or the temperature along the interior curve $T_{calc} = T|_{\Gamma_1}$. Then the solution to the problem may be found by minimising one of the functionals

$$J_1(D) = \|T'_{calc} - q\|_{L^2(\partial\Omega)} \quad (5)$$

or

$$J_2(D) = \|T_{calc} - g\|_{L^2(\Gamma_1)} \quad (6)$$

where g and q are the measured temperature and heat flux. Alternatively, one may use both heat flux and internal temperature measurements and in this case the following functional is minimised

$$J_3(D) = J_1(D) + J_2(D) = \|T'_{calc} - q\|_{L^2(\partial\Omega)} + \|T_{calc} - g\|_{L^2(\Gamma_1)} \quad (7)$$

The domain D can be parameterised in different forms, and the parameters characterising the shape, location and size of the inclusion are determined by minimising one of the functionals (5)-(7). The detection of isotropic and anisotropic inclusions using the functional (5) has been investigated in Mera *et al.*[1]. It is the purpose of this paper to investigate the retrieval of isotropic inclusions using internal measurements, i.e. by minimising the functionals (6) or (7). In this paper we only investigate the cases of circular inclusions, but similar solution methods may be developed for any shape for which the uniqueness of the solution is guaranteed.

3 A real coded genetic algorithm

In this paper, in order to solve the inclusion detection problem we consider a floating point number encoded genetic algorithm similar to the one proposed in Michalewicz [2]. The genetic operators and the parameters used for this genetic algorithm were taken to be population size $n_{pop} = 20$, number of offspring $n_{child} = 30$, uniform arithmetic crossover, crossover probability $p_c = 0.65$, tournament selection, tournament size $k = 2$, tournament probability $p_t = 0.8$, non-uniform

mutation, mutation probability $p_m = 0.5$. We have opted for this GA because, in general, it is known that RCGAs perform better than binary coded GAs for high precision optimisation problems. Moreover, the floating point number encoding of the variables enable natural implementation of fine local tuning processes. Indeed, in this algorithm we use a specialised operator in order to take advantage of the floating point representation of the solution space to increase the rate of convergence of the algorithm, see Mera *et al.* [1].

The inclusion detection problem considered in this paper was reformulated in section 2 as an optimisation problem by minimising one of the functional (5)–(7). Since GAs are, in general, designed to find maxima for functions, the minimum of a functional J was sought as the maximum of the objective function

$$f(D) = \frac{1}{f_{max}^{-1} + J(D)} \quad (8)$$

We note that the maximum of the function f is f_{max} which indicates a perfect fit to the data. This constant was included in the form of the fitness function in order to avoid numerical overflow. A value of $f_{max} = 10^{-2}$ was used for this constant but various values have been used and similar results were obtained.

4 Numerical results

In order to test the convergence and the stability of the method proposed we consider the domain $\Omega = \{(x, y) \mid (x^2 + y^2 < R^2)\}$, i.e. the unit circle, with an isotropic circular inclusion $D = \{(x, y) \mid (x - x_0)^2 + (y - y_0)^2 < r_0^2\}$. In this case, D is parameterised by its location, i.e. its centre (x_0, y_0) , and its size, i.e. its radius r_0 .

In order to calculate the value of the objective functions (5)–(7) for a given possible solution for the inclusion D , an intermediate direct problem of the form (1)–(2) has to be solved. These intermediate direct problems are solved using a Boundary Element Method (BEM), see Brebbia *et al.* [4]. For the problem of inclusion detection, the BEM is particularly suitable since the geometry of the system changes for every possible solution tested during the optimisation process. Further, BEM does not require any meshing of the domain in order to calculate the temperature at various internal points. This reduces the computational effort and eliminates the important perturbations due to changes in the mesh. BEM also reduces the dimensionality of the problem by one by reducing the partial differential equation that governs the process to a boundary integral equation which involves only boundary data. Thus BEM provides clear advantages in comparison with other numerical methods to tackle this kind of inverse geometric problem. Details about the numerical implementation of the BEM to solving direct problems of the form (5)–(7) can be found in Mera *et al.* [1].

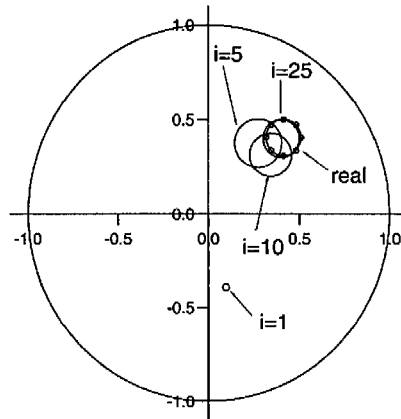


Figure 1: The iterative convergence process as the best solution found by the GA moves toward the exact solution $x_0 = 0.4$, $y_0 = 0.4$, $r_0 = 0.1$ for various numbers of generations performed, namely $i \in \{1, 5, 10, 25\}$, for an isotropic inclusion with $k = 10.0$.

4.1 Retrieval of inclusions with known conductivity

First we investigate the retrieval of an isotropic inclusion with conductivity $k = 10.0$, given by $x_0 = 0.4$, $y_0 = 0.4$ and $r_0 = 0.1$. The inclusion is retrieved by maximising the objective function (6) using $M = 40$ temperature measurements uniformly distributed over the circle of radius $r_1 = 0.8$. The convergence of the iteration process is shown in Figure 1 which presents the numerically retrieved inclusion after performing various numbers of generations. It can be seen that the size and the location of the inclusion D is detected very accurately after about 25 generations.

In order to test the stability of the method, the boundary data given for the heat flux q is perturbed with various amounts of random noise that simulate the inherent measurement errors. Figure 2 presents the numerical solution retrieved for the inclusion D for various levels of noise. It can be seen that as the level of noise decreases, the numerical solution approaches the exact solution while remaining stable. Table 1 presents the values of the parameters x_0 , y_0 , and r_0 characterizing the size and location of the inclusion D for various levels of noise. The results are compared with the corresponding results obtained by using heat flux measurements rather than temperature measurements inside the domain. It can be seen that both methods produce accurate results, although the inversion procedure using heat flux measurements proved to be slightly more stable. Although not presented here, it is reported that numerous other test examples have been investigated and similar results have been obtained.

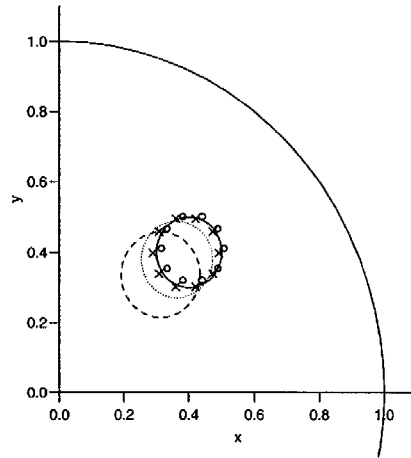


Figure 2: The numerical solution for the cavity D (—) given by $x_0 = 0.4$, $y_0 = 0.4$, $r_0 = 0.1$ for various levels of noise added into the temperature measurements, namely $s = 0\%$ ($\circ$), $s = 1\%$ ($\times$), $s = 2\%$ ($\cdots$) and $s = 3\%$ ($- - -$).

One possible disadvantage of genetic algorithms, in comparison with gradient methods, is the larger amount of computational time required for the genetic search process. However, for the examples considered in this paper it was found that, in general, it is possible to obtain accurate results in as less as 25 generations which involves 750 function evaluations. The BEM is used in order to solve the intermediate direct problems because it does not require any meshing of the domain in order to calculate the boundary data. This reduces the computational effort and eliminates the important perturbations due to changes in the mesh. BEM also reduces the dimensionality of the problem by one by reducing the partial differential equation that governs the process to a boundary integral equation which involves only boundary data. Thus by using a BEM with $N_0 = 40$ and $N_1 = 40$ boundary elements every fitness evaluation takes less than 0.2 seconds CPU time on a Pentium IV processor at 1700Mz. It is worth noting that during the generations many of the individuals generated do not satisfy the constraints and they are assigned a very low fitness value without using the BEM direct solver. Overall, for the examples investigated in this paper it was found that the inclusion can be accurately retrieved in less than 25 generations which were performed in an average of 3 minutes 30 seconds CPU time. Thus, on using genetic algorithms we can obtain more accurate results in a reasonable computational time. It is worth noting that if gradient methods are used then the only way to avoid local convergence is to restart the program from multiple initial guesses and then the user can decide which solution to use. This process can take longer than the time required for one run of the GA which will locate the global optimum.

Table 1: The numerically retrieved values for the parameters x_0 , y_0 , r_0 and k for various levels of noise obtained using interior temperature measurements compared with the corresponding results obtained with heat flux measurements.

Noise		Exact	Temperature measurements	Heat flux measurements
0%	x_0	0.4	0.4106	0.4000
	y_0	0.4	0.4105	0.3999
	r_0	0.1	0.0954	0.1000
1%	x_0	0.4	0.3899	0.4129
	y_0	0.4	0.3983	0.4253
	r_0	0.1	0.1012	0.0946
2%	x_0	0.4	0.3625	0.4255
	y_0	0.4	0.3789	0.4467
	r_0	0.1	0.1089	0.0900
3%	x_0	0.4	0.3132	0.4373
	y_0	0.4	0.3352	0.4641
	r_0	0.1	0.1218	0.0860

4.2 Retrieval of inclusions with unknown conductivity

Next we investigate the possibility of retrieving the size and the location of an isotropic inclusion D when the conductivity k is not known. In this case the conductivity k is considered as a variable in the objective function (8) and the GA is applied to simultaneously determine the size, location and conductivity of the inclusion. Table 2 presents the numerical results obtained for the parameters x_0 , y_0 , r_0 and k for five different GA runs, corresponding to five different sequences of random numbers. We note that in all five runs the parameters x_0 , y_0 and r_0 characterising the size and location of the inclusion D are accurately retrieved. However, for the conductivity k the values obtained by the GA are considerably different for the various runs. We note that this confirms the results obtained in Mera *et al.* [1]. Further it should be noted that for all the solutions presented in Table 2, the fitness function has reached a value close to its maximum and this indicates that convergence has been achieved and no further improvements have been obtained over a larger number of generations. This suggest that the fitness function has multiple local optima corresponding to the multiple solutions retrieved starting with different initial guesses.

Table 2: The numerically retrieved values for the parameters x_0 , y_0 , r_0 and k obtained using interior temperature measurements for five different GA runs.

	<i>exact</i>	<i>run1</i>	<i>run2</i>	<i>run3</i>	<i>run4</i>	<i>run5</i>
x_0	0.4	0.4106	0.4106	0.4106	0.4106	0.4106
y_0	0.4	0.4105	0.4105	0.4105	0.4105	0.4105
r_0	0.1	0.0945	0.0956	0.0925	0.0982	0.0924
k	10.0	11.1297	9.8716	14.5474	7.7791	14.8531

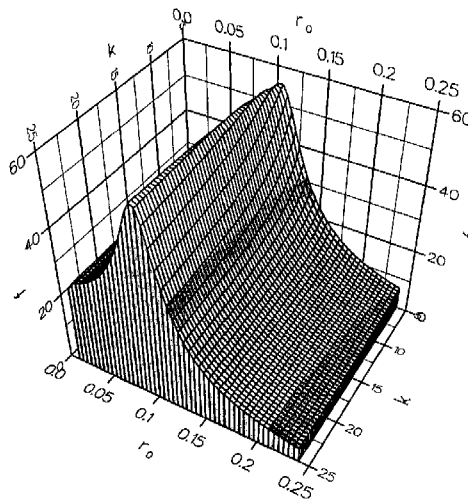


Figure 3: The landscape of the fitness function corresponding to the functional (6) as a function of the parameters r_0 and k for $x_0 = 0.4$ and $y_0 = 0.4$.

The numerical results presented in Table 2 showed that the parameters x_0 , y_0 and r_0 are much easier to identify than the conductivity k . This can be explained by investigating the landscape of the fitness function. Figure 3 presents the fitness function (8) as a function of the parameters r_0 and k if the other two parameters are fixed to $x_0 = 0.4$ and $y_0 = 0.4$. It can be seen that along the line $r_0 = 0.1$ there is a strip parallel to the k axis where the a large numbers of maximizers are located. Due to the steep gradients in the r_0 direction we find relatively easily that r_0 must lie near the value 0.1. However, as Figure 3 suggests it is much more difficult to maximise the objective function along the k direction. Moreover, it was found that the location of the local maximisers along the k axis is very sensitive to small perturbations in the values of the other two parameters x_0 and y_0 . Although not

Table 3: The numerically retrieved values for the parameters x_0 , y_0 , r_0 and k obtained using both interior temperature and boundary heat flux measurements for five different GA runs.

	<i>exact</i>	<i>run1</i>	<i>run2</i>	<i>run3</i>	<i>run4</i>	<i>run5</i>
x_0	0.4	0.4000	0.4000	0.4000	0.3999	0.4000
y_0	0.4	0.4000	0.4000	0.4000	0.4000	0.3999
r_0	0.1	0.0981	0.0974	0.1041	0.1039	0.1006
k	10.0	12.2580	13.4379	7.1324	7.2162	9.3753

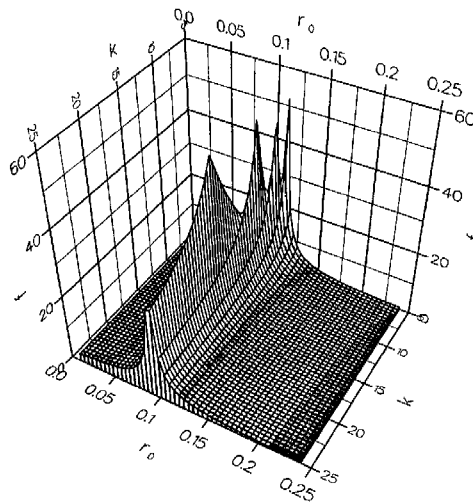


Figure 4: The landscape of the fitness function corresponding to the functional (7) as a function of the parameters r_0 and k for $x_0 = 0.4$ and $y_0 = 0.4$.

presented here, it is reported that similar results are obtained if the fitness function is plotted as a function of the variables x_0 and k or y_0 and k . This explains why the parameters x_0 , y_0 and r_0 are accurately retrieved in the case when the conductivity k is not known, while the conductivity k cannot be retrieved.

It worth noting that similar results have been obtained if heat flux measurements are used and the objective function (5) is optimised, see Mera *et al.* [1]. Therefore, next we investigate the results obtained by using both interior temperature and heat flux measurements, by optimising the objective function (7). The heat flux is specified at $N_0 = 40$ points on the boundary and the temperature is specified at the same points on the boundary and at $M = 40$ points inside the domain, on the circle of radius $r_1 = 0.8$. Table 3 presents the numerical results obtained for

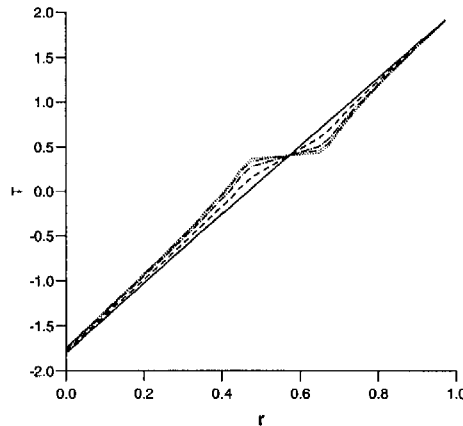


Figure 5: The temperature along the radius given by $\theta = \pi/4$ calculated using various conductivities, namely, $k = 1.0$ (—), $k = 2.0$ (---), $k = 5.0$ (- · -), $k = 10.0$ (- · · · -) and $k = 20.0$ (···).

the parameters x_0 , y_0 , r_0 and k for five different GA runs, corresponding to five different sequences of random numbers. We note that also in this case in all five runs the parameters x_0 , y_0 and r_0 characterising the size and location of the inclusion D are accurately retrieved. It worths noting that if both heat flux and interior temperature measurements are used then the results obtained for the parameters x_0 , y_0 and r_0 are more accurate approximations for their real values, see Table 3, when compared with the results obtained with only temperature measurements, see Table 2. However, for the conductivity k the values obtained by the GA are considerably different for the various runs. The fitness function (8) has a similar shape to the one obtained when using only temperature measurements. Also in the case of using both heat flux and temperature measurements, the fitness function landscape contains a strip parallel to the k axis where a large numbers of local maximizers are located, see Figure 4. This explains why the parameters x_0 , y_0 and r_0 are accurately retrieved in the case when the conductivity k is not known, while the conductivity k cannot be retrieved, even if the problem is overspecified by imposing a large number of temperature and heat flux measurements. The presence of more maximisers in the strip along the k axis suggests that for the problem of determining the conductivity k from heat flux or temperature measurements uniqueness does not hold, at least from a numerical point of view.

Figure 5 presents the numerical solution for the temperature along the radius that passes through the centre of the cavity D obtained by using various values for the conductivity k in the direct problem (1)–(2). It can be seen that large changes in the thermal conductivity only produce small perturbations in the temperature in the regions next to the inclusion D given by $x_0 = 0.4$, $y_0 = 0.4$, $r_0 = 0.1$. These changes in the conductivity produce negligible perturbations in the boundary data

or in the temperature at points situated not in the immediate vicinity of D . Since these small perturbations are dominated by noise coming from the measurement errors it follows that the thermal conductivity is very difficult to retrieve accurately. However, even if the thermal conductivity is not known, the inversion procedure proposed in this paper is very efficient in retrieving the size and the position of the inclusion.

5 Conclusions

In this paper the inverse conductivity problem which requires the determination of the location, size and/or non-dimensional isotropic conductivity, k , of a circular inclusion D contained in a domain Ω from temperature measurements on the boundary $\partial\Omega$ and in the interior of the domain Ω has been investigated numerically using a GA. It was found that the retrieval of the location and size of the circular inclusion is convergent and stable even if the thermal conductivity is not known. However, it was found that the thermal conductivity cannot be stably retrieved from boundary and/or interior data. Overall, it may be concluded that the GA based method developed in this paper is a robust, efficient method for detecting the size and location of subsurface inclusions.

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