Comparing the conventional displacement, first-kind and second-kind BIEs in frictionless contact problems

R. Vodička¹, A. Blázquez², F. Paris³ and V. Mantić³
¹Faculty of Mechanical Engineering, Tech. Univ. of Košice, Slovakia
²Dept. of Mechanical Engineering, Univ. of La Rioja, Spain
³School of Engineering, Univ. of Seville, Spain

Abstract

There are several formulations of Boundary Integral Equations (BIEs) used in the general numerical procedure known as Boundary Element Method (BEM). There are also several approaches to deal with contact problems in BEM. In this paper a comparison between several procedures: conventional discretization of the displacement BIE by collocations, Galerkin discretizations of the symmetric BIE of the first kind and the non-symmetric BIE of the second kind, is performed. Although several aspects of these procedures are discussed, the emphasis is placed on the accuracy of the results obtained with identical meshes. The comparison is carried out including problems with analytical solutions or in presence of singularities, covering conforming, advancing and receding contact problems. Linear elements, conforming discretizations of surfaces in contact and frictionless case define the frame in which the study is performed.

1 Introduction

The aim of the present paper is to contribute to the development and understanding of different possibilities offered by BEM for the solution of contact problems. A numerical comparative study of three basic variants of BIEs of the so-called direct BEM applied to the solution of two-dimensional frictionless contact problems is presented in this paper.

The first BIE studied here is the one traditionally used in BEM applications, defined by the Somigliana displacement identity, denoted here as u-BIE. The other two BIEs examined here, sometimes called coupled BIEs, are obtained by a combination of Somigliana displacement and traction identities, the latter being
denoted here as $t$-BIE. The choice of a BIE applied at a point of the boundary depends on the kind of boundary condition prescribed at this point. The first-kind symmetric BIE is obtained when $u$-BIE is applied on Dirichlet part and $t$-BIE on Neumann part of the boundary, and vice-versa for the second-kind BIE.

It should be pointed out that each of these BIEs has some advantage over the other two. Some of these advantages are well known, like simplicity of implementation of $u$-BIE in a BEM code, symmetry of the linear system obtained by a Galerkin discretization of the first-kind BIE (usually termed Symmetric Galerkin BEM - SGBEM) and the lowest condition number of the linear system obtained by a collocation discretization of the second-kind BIE.

In order to maintain an advantageous characteristic of the coupled BIEs, namely symmetry and the second-kind character of the linear operators associated to the first- and second-kind BIEs respectively, a special technique of equation assembly is required. Vodička [1] introduced a transformation of contact variables which yields a required assembly technique for solution of frictionless contact problems by coupled BIEs.

For solution of contact problems, a BIE has to be coupled with nonlinear contact conditions. A direct imposition of contact conditions, originally developed for $u$-BIE by Andersson et al. [2] is applied for all BIEs studied in the present work. Contact conditions can be imposed either in a strong sense using a node-to-node scheme, for conforming meshes in contact, and a node-to-point scheme, for non-conforming meshes (Paris et al. [3]), or in a weak sense (Blázquez et al. [4]). It can be proved that for conforming meshes, with a one-to-one correspondence of nodes in both meshes in contact, the strong node-to-node and weak approaches are in fact equivalent.

Numerical solutions of three typical contact problems obtained by the above three basic BIE approaches are presented and analysed in the present work. The comparison between the three BIE approaches is facilitated by using for each problem a unique discretization, with conforming meshes in contact.

BEM codes applied in this study use discretizations by continuous linear boundary elements. Principal features of the BEM codes used are the following: i) $u$-BIE approach - collocations of $u$-BIE at mesh nodes, numerical integrations using 8 Gauss points, and an imposition of contact conditions in a weak sense; ii) coupled BIE approaches - Galerkin discretization, analytical integrations and a strong imposition of contact conditions.

2 Applications

2.1 Indentation of a cylinder against a foundation

The geometry, properties and boundary conditions are defined in Fig. 1. The concentrated load is applied by means of a triangular distribution along the two elements closest to the axis $y$ of symmetry. The presence of such an axis permits the symmetry to be applied. The coupled BIE approach uses it explicitly (putting elements along the axis of symmetry), whereas in the $u$-BIE approach explicit and implicit (not putting elements along the axis of symmetry but integrating along the whole boundary) symmetry are applied, the second case being to all effects comparable, in terms of results, to not considering the symmetry.
The load, as indicated in Fig. 1, is applied by means of a parameter $\lambda$, which varies from 0 to 1. The contact algorithms associated to all approaches are able to look for the best solution (length of the contact zone), in accordance with the discretization performed, associated to a value of $\lambda$, applying a trial and error procedure. The contact algorithm used in $u$-BIE approach is additionally prepared to detect the value of $\lambda$ required to reach the contact at any node of the contact zone discretization performed. The zone candidate to reach contact is modelled with elements of identical length of value 0.00225 mm.

Fig. 2 represents the values of the contact pressure along the contact zone for five different values of $\lambda$ (0.05, 0.25, 0.5, 0.75, 1.0). As can be observed, the results agree very well both with each other and with the Hertz solution except for some cases, at the axis of symmetry and at the extreme of the contact zone.

At the axis of symmetry the errors that can be considered significant are only associated, as can be more clearly observed in Fig. 3 where only the maximum value of the contact pressure is represented for different values of $\lambda$, to the use of explicit symmetry for the $u$-BIE approach. These errors can be associated to the presence of mixed boundary conditions at the artificial corner originated by the application of the explicit symmetry and are not associated to the contact problem formulation or solution. Nevertheless, it is possible to observe the beneficial effect on the accuracy at corners with mixed boundary conditions that Galerkin discretization has in comparison to the collocation discretization.

With reference to the errors close to the end of the contact zone, clearly noticeable in Fig. 2, they are of a different nature.

In Fig. 2 a particular case of $\lambda=0.3628$ that produces the incorporation of a node to the contact zone has been included (for clarity only for $u$-BIE approach). It can be seen that the numerical contact zone is slightly bigger than the analytical contact zone for the load considered. In terms of contact pressure, only a certain error appears at the first node inside the contact zone, with a value slightly higher than the analytical. The reason for this small error is that the linear element used is not able to capture the infinite value of the slope of the stresses at the onset of the contact zone, a very local error compensation appearing at the neighbourhood of the end contact zone to guarantee the global
equilibrium of each solid.

When the value of $\lambda$, as is the case for the other values selected and represented in Fig. 2, corresponds to an intermediate value between those corresponding to the incorporation to the contact zone of two consecutive nodes, the contact pressure at the final node of the contact zone is not zero. In Fig. 2 the position of the following node not yet in contact, and consequently with zero
value of the contact pressure, has also been represented, due to the fact that along the element defined by these two nodes there is a linear distribution of load due to the approximation performed, although the element is not strictly in contact. The correct value of the contact zone would be somewhere in the middle of this element. It can be observed that the errors associated to the lack of accuracy of the discretization used for the particular value of $\lambda$ considered (see e.g. $\lambda = 1$) are balanced very locally with respect to the analytical solution, involving the first node (the first two at most) closest to the end of the contact zone.

All previously stated for the length of the contact zone can be observed in Fig. 4, where the half length of the contact zone is represented versus $\lambda$. $u$-BIE predictions for the values of $\lambda$ that produce the incorporation of a new node to the contact zone present the small differences previously mentioned, whereas pre-fixed values of $\lambda$ may lead to certain differences in predictions (presented for the coupled BIE approaches) depending on the discretization performed.

2.2 Compression of a thin layer on a foundation

The geometry, properties, loads and boundary conditions are defined in Fig. 5. The concentrated load is applied in a similar manner to the previous problem. All results shown correspond to considering implicit symmetry.

The nature of the problem leads to a final length of the contact zone smaller than the original but independent of the amount of load applied, that indicated in Fig. 5 then being applied in one increment. The length of the contact zone is controlled by the value of the first Dundurs parameter $\alpha$ (Keer et al. [5]).

Fig. 6 represents the distributions of the contact pressures along the contact zone with the three procedures for $\alpha = 0$. The maximum values of the contact pressure for the whole interval of possible values of $\alpha$ (between 1 and -1) are represented in Fig. 7.

The evolution of the size of the contact zone with the variation of the relative
stiffness of layer and foundation can be observed in Fig. 8. The analytical solution of Keer et al. [5] for the case of infinite domains is also represented in this figure. Results for the second-kind BIE are omitted for $\alpha<0$ in Fig. 8 due to inaccurate evaluation of displacements in these cases, although, as can be seen in Fig. 7, maximum pressure obtained is reasonably accurate.

It can be observed that for $\alpha>0$ the predictions of the three numerical approaches coincide with each other and with Keer et al. solution. In contrast, for negative values of $\alpha$ (the layer is stiffer than the foundation), some differences in the numerical predictions appear, these not always being close to Keer et al. solution. It could be thought that these discrepancies are due to the different

![Figure 5: Receding contact problem configuration.](image)

![Figure 6: Contact pressure distribution.](image)
adaptation of the mesh (common for coherence to all cases independently of the value of $\alpha$) to the nature of the problem. Thus, for $\alpha > 0$ the contact zone is very small and the mesh, prepared to capture the contact zone in this case, was refined only in the expected contact zone.

When a fine mesh has been used (results not presented here) it has been observed that, although the solution has slightly improved at the end of the contact zone, capturing the solution in a smoother way, it does not significantly alter the former numerical solution. Having arrived at this point, it should be observed that the numerical solution obtained for the problem modelled is correct and that the discrepancies with respect to the analytical solution are in fact due to the different nature of the problems solved analytically (an infinite layer) and numerically (a finite length of the layer) when $\alpha$ tends to -1.

2.3 Indentation of a punch against a foundation

The geometry, properties, loads and boundary conditions are defined in Fig. 9. All results shown correspond to considering explicit symmetry.

The nature of the problem leads to a length of the contact zone independent of the amount of load applied and coincident with the original contact zone. The distribution of pressures along the contact zone is represented in Fig. 10, where a detail close to the corner is represented to observe more clearly the effects on the results of the presence of a singularity in the stresses.

It can immediately be seen that the results are almost indistinguishable in the three approaches along the whole contact zone out of the area dominated by the singularity, where $u$-BIE discretized by collocations is the most sensible approach whereas the Galerkin discretization of the coupled BIEs smoothes out to some extent the perturbations in the results originated by the singularity.

![Figure 7: Contact pressure at the axis of symmetry.](image-url)
It is suitable to evaluate how the three approaches estimate the representative parameters of the singular stresses. To this end, the following expression for the contact pressure at the neighbourhood of the corner is assumed:

$$\sigma \equiv K (2\pi r)^{-\lambda}$$  \hspace{1cm} (1)

where $\sigma$ is the contact pressure at a point placed at a distance $r$ from the corner, $\lambda$ is the order of the singularity and $K$ is the generalized stress intensity.
Figure 10: Singular contact pressure distribution.

factor. The local solution at the neighbourhood of the corner is known, Dundurs and Lee [6], presenting a value of \( \lambda = 0.2260 \).

Tab. 1 shows the results obtained for \( K \), and also those predicted for \( \lambda \), in order to compare with the analytical value, by means of a least square approach considering \(-1.75 < \log r < 0.1\).

Table 1: Singularity exponent and generalized stress intensity factor evaluation.

<table>
<thead>
<tr>
<th>( u \text{-BIE} )</th>
<th>first-kind BIE</th>
<th>second-kind BIE</th>
</tr>
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<tbody>
<tr>
<td>( \lambda ) (error)</td>
<td>0.2354 (4%)</td>
<td>0.2266 (0.27%)</td>
</tr>
<tr>
<td>( K )</td>
<td>168.91</td>
<td>168.80</td>
</tr>
</tbody>
</table>

Although the results are in all cases quite accurate in evaluating \( \lambda \), it can be seen again that the coupled BIEs discretized by Galerkin approach is better equipped to deal with the presence of singularities. \( u \text{-BIE} \) discretized by collocations would require a finer mesh for the case analysed.

3 Conclusions

Accuracy of numerical solutions of frictionless contact problems obtained by the symmetric first-kind BIE (SGBEM) and the second-kind BIE has been compared with the solution by the conventional \( u \text{-BIE} \). In this study it has been shown that solution by the first- and second-kind BIEs discretized by Galerkin approach can be competitive with solution of such problems by conventional collocation discretization of \( u \text{-BIE} \). In some configurations, with corners or stress singularities, these novel approaches can even present some advantages.
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References


