Dual reciprocity formulation for elasticity problems using compact supported radial basis functions

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Abstract

In this paper particular solutions corresponding to the compact supported radial basis functions (CS RBF) to the Navier operator of elasticity are derived. The use of these particular solutions within the dual reciprocity method (DRM) is outlined. The fictitious displacement and traction particular solution kernels are derived for four classes of CS RBF. Additional complementary solutions are derived to replace the particular solutions outside the zone of the compact radius (α). The continuity of the two solutions and their derivatives up to the third derivatives are ensured along the circumference of the compact supported edge circle. All kernels are listed in explicit forms.

1 Introduction

The Dual Reciprocity Method (DRM), developed firstly by Nardini and Brebbia [1], provides an efficient solution for engineering applications governed by inhomogeneous differential equations. The main advantage of the DRM is the boundary-only discretisation; even for problems having domain-type terms. The DRM has been used to solve potential, fluid mechanics and heat transfer problems [2], general body forces and dynamic problems [3,4].

The main idea of the DRM is to treat the inhomogeneous terms as pseudo body forces, which are represented by another integral identity. Both the original and the body force integral formulation use the Green reciprocity theory; therefore the formulation is said to be dual reciprocity formulation. In order to manipulate the inhomogeneous terms, a collocation is carried out to interpolate such forces as continuous surface. This step could be done using Radial Bases Functions (RBF). Most of the DRM development until recently uses the ad-hoc (1+R) basis function in the approximation of the inhomogeneous terms. Golberg and Chen
[5] studied different types of RBF. They showed that the (1+R) function is just one type of a class of RBF called conical. They also presented alternative classes of RBF, such as splines, multi-quadratic, etc. Ref. [5] presents a survey of these functions.

It is known that most of the RBF are globally defined basis functions. This means the resulting matrix for interpolation is dense and can be highly ill conditioned, especially for a large number of interpolation points. This poses serious stability problems and high computational cost. All of these drawbacks lead the researchers to search for basis functions that have local support. The most popular compact supported RBF (CS RBF) are the ones constructed by Wendland [6], Chen et al. [7] implemented the CS RBF with the DRM to solve potential problems. Golberg et al. [8] used similar formulation for 3D problems and Cheng et al. [9] studied the performance of CS RBF with iterative solvers. However, to the author's best knowledge the application of the CS RBF to elasticity equations via the DRM has never been reported previously.

In this paper particular solutions corresponding to the compact supported radial basis functions (CS RBF) to the Navier operator of elasticity are derived. All kernels are listed in explicit forms.

## 2 Dual Reciprocity Formulation

For a general body having a domain \( \Omega \) with boundary \( \Gamma \), if \( u_j \) & \( t_j \) denote the boundary displacement & traction vectors respectively and \( b_i \) denote the body forces or term that defines inhomogeneous terms, the corresponding boundary integral equation can be defined as follows [10]:

\[
C_{ij}u_j + \int_{\Gamma} T_{ij}u_j d\Gamma - \int_{\Gamma} U_{ij} t_j d\Gamma = \int_{\Omega} U_{ij} b_j d\Omega
\]  

(1)

where, \( C_{ij} \) denotes the jump terms. The \( U_{ij} \) & \( T_{ij} \) denote the two-point fundamental solution kernels. If the same problem is re-defined using a collocation based field of displacements and tractions \( \Psi_{jm} \) and \( \eta_{jm} \) respectively, Eq. (1) can be redefined as follows:

\[
\begin{aligned}
\left[ C_{ij}\Psi_{jm} + \int_{\Gamma^*} T_{ij}\Psi_{jm} d\Gamma^* - \int_{\Gamma^*} U_{ij}\eta_{jm} d\Gamma^* \right] \alpha_m &= \int_{\Omega^*} U_{ij} B_j d\Omega^* \\
\end{aligned}
\]  

(2)

where the domain \( \Omega^* \) and its boundary \( \Gamma^* \) can be chosen arbitrary and \( B_j \) is the new body force term that generates the displacements and tractions \( \Psi_{jm} \) and \( \eta_{jm} \), and \( \alpha_m \) is the coefficient of collocation which can be defined as:

\[
B_j = f \alpha_j
\]  

(3)

where \( f \) can be chosen as arbitrary function. If the \( \Omega^* \) and \( \Gamma^* \) are chosen to be \( \Omega \) and \( \Gamma \) and \( \alpha_i \) is computed assuming that \( B_j \) is equal to \( b_i \), one can use both Eqs (1) and (2) to solve the problem with boundary-only integrals. Eqs (1) and (2)
which are based on the Green’s reciprocal theory, are the basic equations for the dual reciprocity formulation.

In order to use Eq. (2), the expressions for the fictitious displacements and tractions fields should be derived first. It has been proven by Rashed [4] that such fields can be expressed in terms of the fictitious Galerkin tensor \( g \) as follows:

\[
\Psi_{ij} = \frac{1}{\mu} \delta_{ij} \left[ \frac{g'}{R} + g'' \right] - \frac{1}{2(1-\nu)\mu} \left[ \frac{g'}{R} \left( \delta_{ij} - R_{ii} R_{jj} \right) + g'' R_{ii} R_{jj} \right]
\]  

(4)

and

\[
\eta_{ij} = N_{i} R_{ii} \left[ \frac{\nu}{1-\nu} \left( \frac{g'}{R} \right) + g''' \right] + N_{i} R_{ij} \left[ \left( \frac{g'}{R} \right)^2 - \frac{g''}{R} \right] + \frac{\nu}{1-\nu} g'''
\]  

(5)

where \( \mu \) denotes the shear modulus and \( \nu \) denotes the Poisson’s ratio, \( R \) is the Euclidian distance between the field and the source points and

\[
\nabla^4 g = f
\]  

(6)

in which \( \nabla^4 \) is the two dimensional bi-harmonic operator. The expressions for \( g \) and its relevant derivatives are derived in section 4.

1 Compact Supported Radial Basis Functions

The function \( f \) in Eq. (6) can be selected to be any RBF. In this paper \( f \) will be chosen as CS RBF as given in Table 1 (see Wendland [6]).

2 Relevant Derivatives

In this section the expressions for the fictitious Galerkin tensor \( g \) and its derivatives will be derived for the four CS RBF listed in the former section:

2.1 The First Function

The expression for \( g \) can be obtained as follows:

\[
g = \frac{R^4}{64} - \frac{2R^5}{225\alpha} + \frac{R^6}{576\alpha^2}
\]  

(7)

The corresponding relevant derivatives can be computed as follows:

\[
g' = \frac{R^3}{1440\alpha^2} \left( 90\alpha^2 - 64\alpha R + 15R^2 \right)
\]  

(8)

\[
g'' = \frac{R^2}{1440\alpha^2} \left( 270\alpha^2 - 256\alpha R + 75R^2 \right)
\]  

(9)
\[ g''' = \frac{R}{120 \alpha^2} \left( 45 \alpha^2 - 64 \alpha R + 25 R^2 \right) \]  

(10)

Table 1: The used CS RBF.

<table>
<thead>
<tr>
<th>Number</th>
<th>Function</th>
<th>Continuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>( f = \left( 1 - \frac{R}{\alpha} \right)^2 )</td>
<td>( C^{(0)} )</td>
</tr>
<tr>
<td>Second</td>
<td>( f = \left( 1 - \frac{R}{\alpha} \right)^4 \left( 1 + \frac{4 R}{\alpha} \right) )</td>
<td>( C^{(2)} )</td>
</tr>
<tr>
<td>Third</td>
<td>( f = \left( 1 - \frac{R}{\alpha} \right)^6 \left( 3 + 18 \frac{R}{\alpha} + 35 \left( \frac{R}{\alpha} \right)^2 \right) )</td>
<td>( C^{(4)} )</td>
</tr>
<tr>
<td>Fourth</td>
<td>( f = \left( 1 - \frac{R}{\alpha} \right)^8 \left( 1 + \frac{8 R}{\alpha} + 25 \left( \frac{R}{\alpha} \right)^2 + 32 \left( \frac{R}{\alpha} \right)^3 \right) )</td>
<td>( C^{(6)} )</td>
</tr>
</tbody>
</table>

The symbol (Expression)\(^{+}\) means \( f = \text{Expression} \); if \( R < \alpha \) and \( f = 0 \); if \( R > \alpha \), and \( \alpha \) denotes the radius of the compact support.

2.2 The Second Function

The expression for \( g \) can be obtained as follows:

\[ g = \frac{R^4}{64} - \frac{5R^6}{288 \alpha^2} + \frac{4R^7}{245 \alpha^3} - \frac{5R^8}{768 \alpha^4} + \frac{4R^9}{3969 \alpha^5} \]  

(11)

The corresponding relevant derivatives can be computed as follows:

\[ g' = \frac{R^3}{70560 \alpha^5} \left( 4410 \alpha^5 - 7350 \alpha^3 R^2 + 8064 \alpha^2 R^3 - 3675 \alpha R^4 + 640 R^5 \right) \]  

(12)

\[ g'' = \frac{R^2}{70560 \alpha^5} \left( 13230 \alpha^5 - 36750 \alpha^3 R^2 + 48384 \alpha^2 R^3 - 25725 \alpha R^4 + 5120 R^5 \right) \]  

(13)

\[ g''' = \frac{R}{1008 \alpha^5} \left( 378 \alpha^5 - 2100 \alpha^3 R^2 + 3456 \alpha^2 R^3 - 2205 \alpha R^4 + 512 R^5 \right) \]  

(14)

2.3 The Third Function

The expression for \( g \) can be obtained as follows:
The corresponding relevant derivatives can be computed as follows:

\[ g = \frac{3R^4}{64} - \frac{7R^6}{144\alpha^2} + \frac{35R^8}{384\alpha^4} - \frac{64R^9}{567\alpha^5} + \frac{21R^{10}}{320\alpha^6} - \frac{64R^{11}}{3267\alpha^7} + \frac{7R^{12}}{2880\alpha^8} \]  

\[ g' = \frac{R^3}{332640\alpha^8} \left( 62370\alpha^8 - 97020\alpha^6 R^2 + 242550\alpha^4 R^4 - 337920\alpha^3 R^5 + 218295\alpha^2 R^6 - 71680\alpha R^7 + 9702 R^8 \right) \]  

\[ g'' = \frac{R^2}{332640\alpha^8} \left( 187110\alpha^8 - 485100\alpha^6 R^2 + 1697850\alpha^4 R^4 - 2703360\alpha^3 R^5 + 1964655\alpha^2 R^6 - 716800\alpha R^7 + 106722 R^8 \right) \]  

\[ g''' = \frac{R}{792\alpha^8} \left( 891\alpha^8 - 4620\alpha^6 R^2 + 24255\alpha^4 R^4 - 45056\alpha^3 R^5 + 37422\alpha^2 R^6 - 15360\alpha R^7 + 2541 R^8 \right) \]

2.4 The Fourth Function

The expression for \( g \) can be obtained as follows:

\[ g = \frac{R^4}{64} + \frac{11R^6}{576\alpha^2} + \frac{11R^8}{384\alpha^4} + \frac{231R^{10}}{3200\alpha^6} + \frac{32R^{11}}{297\alpha^7} + \frac{77R^{12}}{960\alpha^8} + \frac{64R^{13}}{1859\alpha^9} + \frac{11R^{14}}{1344\alpha^{10}} + \frac{32R^{15}}{38025\alpha^{11}} \]  

(19)

The corresponding relevant derivatives can be computed as follows:

\[ g' = \frac{R^3}{16061760\alpha^{11}} \left( 1003860\alpha^{11} - 1840410\alpha^9 R^2 + 3680820\alpha^7 R^4 - 11594583\alpha^5 R^6 + 19036160\alpha^2 R^7 - 15459444\alpha^3 R^8 + 7188480\alpha^2 R^9 - 1840410\alpha R^{10} + 202752 R^{11} \right) \]  

(20)

\[ g'' = \frac{R^2}{16061760\alpha^{11}} \left( 3011580\alpha^{11} - 9202050\alpha^9 R^2 + 25765740\alpha^7 R^4 - 104351247\alpha^5 R^6 + 190361600\alpha^4 R^7 - 170053884\alpha^3 R^8 + 86261760\alpha^2 R^9 - 23925330\alpha R^{10} + 2838528 R^{11} \right) \]  

(21)

\[ g''' = \frac{R}{1560\alpha^{11}} \left( 585\alpha^{11} - 3575\alpha^9 R^2 + 15015\alpha^7 R^4 - 81081\alpha^5 R^6 + 166400\alpha^4 R^7 - 165165\alpha^3 R^8 + 92160\alpha^2 R^9 - 27885\alpha R^{10} + 3584 R^{11} \right) \]  

(22)
Figure 1: Particular and complementary solutions.

3 Associate Complementary Solutions

The expressions for $g$ are suitable for all points located within the compact support radius. Provided that the value of the RBF is zero outside this region (see figure 1), it is important to use alternative expression for $g$ that satisfies:

$$\nabla^4 g = 0 \quad \text{where} \quad \frac{R}{\alpha} > 1$$

(23)

In this case $g$ can be chosen as a set of complementary solutions for the bi-harmonic operator. It is important to satisfy the continuity for $g$, $g'$, $g''$, $g'''$ at the circumference circle; where $R=\alpha$. In order to do so, $g$ has to be chosen as a combination of at least four complementary solutions for the bi-harmonic operators to allow the existence of the derivatives up to the third one, therefore:

$$g = a R^2 \ln R + b R^2 + c \ln R + d$$

(24)

and its relevant derivatives:

$$g' = a (R + 2R \ln R) + 2 b R + \frac{c}{R}$$

(25)

$$g'' = a (3 + 2 \ln R) + 2 b - \frac{c}{R^2}$$

(26)
where \( a, b, c, d \) are arbitrary constants to be determined using the continuity conditions. By matching the values of \( g \) and its derivatives in sections 4 and 5 at \( R=\alpha \), the following expressions are obtained:

\[
a = \frac{-1}{4\alpha} g'(\alpha) + \frac{1}{4} g''(\alpha) + \frac{\alpha}{4} g'''(\alpha)
\]

\[
b = \frac{8 + 4 \ln \alpha}{\alpha} g'(\alpha) - \frac{\ln \alpha}{4} g''(\alpha) - \frac{\alpha(1 + \ln \alpha)}{4} g'''(\alpha)
\]

\[
c = \frac{\alpha}{4} g'(\alpha) - \frac{\alpha^2}{4} g''(\alpha) + \frac{\alpha^3}{4} g'''(\alpha)
\]

\[
d = g(\alpha) \left( \frac{\alpha(2 + \ln \alpha)}{4} g'(\alpha) + \frac{\alpha^2 \ln \alpha}{4} g''(\alpha) - \frac{\alpha^3 (1 - \ln \alpha)}{4} g'''(\alpha) \right)
\]

where \( g(\alpha), g'(\alpha), g''(\alpha), g'''(\alpha) \) are the values of \( g, g', g'', g''' \) when \( R=\alpha \) (recall section 4). The following sections give the values of \( g(\alpha), g'(\alpha), g''(\alpha), g'''(\alpha) \) and the corresponding constants \( a, b, c, d \) for each of the previously considered four CS RBF.

### 3.1 The First Function

From Eqs. (7-10), substituting with \( R=\alpha \), it gives:

\[
g(\alpha) = \frac{61\alpha^4}{7200} \quad g'(\alpha) = \frac{41\alpha^2}{1440} \quad g''(\alpha) = \frac{89\alpha^2}{1440} \quad g'''(\alpha) = \frac{\alpha}{20}
\]

Substituting in Eqs. (28-31), it gives:

\[
a = \frac{\alpha^2}{48} \quad b = \frac{\alpha^2 (1 - 12 \ln \alpha)}{576} \quad c = \frac{\alpha^4}{240} \quad d = \frac{\alpha^4 (97 - 60 \ln \alpha)}{14400}
\]

### 3.2 The Second Function

From Eqs. (11-14), substituting with \( R=\alpha \), it gives:

\[
g(\alpha) = \frac{46169\alpha^4}{5080320} \quad g'(\alpha) = \frac{2089\alpha^3}{70560} \quad g''(\alpha) = \frac{4259\alpha^2}{70560} \quad g'''(\alpha) = \frac{41\alpha}{1008}
\]

Substituting in Eqs. (28-31), it gives:
3.3 The Third Function

From Eqs. (15-18), substituting with \( R = \alpha \), it gives:

\[
g(\alpha) = \frac{109774 \alpha^4}{43908480} \quad g'(\alpha) = \frac{26297 \alpha^3}{332640} \quad g''(\alpha) = \frac{51077 \alpha^2}{332640} \quad g'''(\alpha) = \frac{73 \alpha}{792}
\]

(35)

Substituting in Eqs. (28-31), it gives:

\[
a = \frac{\alpha^2}{24} \quad b = \frac{\alpha^2(997 - 2520 \ln \alpha)}{60480} \quad c = \frac{7 \alpha^4}{1584} \quad d = \frac{7 \alpha^4(7631 - 3960 \ln \alpha)}{6272640}
\]

(36)

3.4 The Fourth Function

From Eqs (19-22), substituting with \( R = \alpha \), it gives:

\[
g(\alpha) = \frac{855163 \alpha^4}{112432320} \quad g'(\alpha) = \frac{75445 \alpha^3}{3212352} \quad g''(\alpha) = \frac{706697 \alpha^2}{16061760} \quad g'''(\alpha) = \frac{19 \alpha}{780}
\]

(37)

Substituting in Eqs. (28-31), it gives:

\[
a = \frac{7 \alpha^2}{624} \quad b = \frac{7 \alpha^2(25943 - 51480 \ln \alpha)}{32123520} \quad c = \frac{\alpha^4}{1040} \quad d = \frac{\alpha^4(48791 - 24024 \ln \alpha)}{24984960}
\]

(38)

4 Expressions For Fictitious Displacement and Traction Fields \( \Psi_{ij} \) and \( \eta_{ij} \)

After obtaining the appropriate expressions for \( g \), it is easy to obtain the expressions for \( \Psi_{ij} \) and \( \eta_{ij} \) by substituting in Eqs. (4-5).

4.1 The First Function

For \( R \leq \alpha \), substitute from Eqs. (8-10) into (4-5), it gives:

\[
\Psi_{ij} = \frac{1}{(1 - \nu)\mu} \left( \frac{R}{\alpha} \right)^k \frac{\delta_{ij}}{2880} \left( 90(7 - 8\nu)\alpha^2 - 64\nu \alpha R + 15(11 - 12\nu)R^2 \right) - \frac{R_{ij} R_{ij}}{240} \left( \frac{R}{\alpha} \right) \left[ 15 \alpha^2 - 16 \alpha R + 5 R^2 \right]
\]

(40)
\[ \eta_{ij} = \frac{R}{120 \alpha^2} \left( -15 \alpha^2 + 16 \alpha R + 5 R^2 \right) \times \]
\[
\left[ \frac{1}{(1 - \nu)} \left( \nu (N_j R_{si} + \delta_{ij} R_{sN}) - 3 R_{si} R_{sj} + N_j R_{sj} \right) + \right.
\]
\[
\left. \left( 45 \alpha^2 - 64 \alpha R + 25 R^2 \right) \left[ \frac{1}{(1 - \nu)} \left( \nu N_i R_{ij} - R_{si} R_{sj} + N_i R_{sj} + \delta_{ij} R_{sN} \right) \right] \right] \]
\]
\[ (41) \]

For \( R \geq \alpha \), substitute from Eq. (33) into Eqs. (25-27), then into Eqs. (4-9), it gives:
\[ \Psi_{ij} = \frac{1}{(1 - \nu) \mu} \left\{ \delta_{ij} \left[ \frac{\alpha^2}{48} \left( 1 + \ln R \right) + \frac{\alpha^2}{576} \left( 1 - 12 \ln \alpha \right) \right] (7 - 8 \nu) \right. \]
\[
\left. \left[ \frac{\alpha^2}{576} \left( 1 - 12 \ln \alpha \right) + \frac{\alpha^4}{240 R^2} \right] - R_{si} R_{sj} \left[ \frac{\alpha^2}{48} - \frac{\alpha^4}{240 R^2} \right] \right\} \]
\[ (42) \]

\[ \eta_{ij} = \frac{\alpha^2}{24 R} \left\{ -1 + \frac{\alpha^2}{5 R^2} \right\} \left[ \frac{1}{(1 - \nu)} \left( \nu (N_j R_{si} + \delta_{ij} R_{sN}) - 3 R_{si} R_{sj} + N_j R_{sj} \right) + \right. \]
\[
\left. \left[ 1 + \frac{\alpha^2}{5 R^2} \right] \left[ \frac{1}{(1 - \nu)} \left( \nu N_i R_{ij} - R_{si} R_{sj} + N_i R_{sj} + \delta_{ij} R_{sN} \right) \right] \right\} \]
\[ (43) \]

4.2 The Second Function

For \( R \leq \alpha \), substitute from Eqs. (12-14) into (4-5), it gives:
\[ \Psi_{ij} = \frac{R^2}{24(1 - \nu) \mu \alpha^5} \left\{ 24(1 - \nu) \times \right. \]
\[
\left. \left( 1470 \alpha^5 - 365 \alpha^3 R^2 + 4704 \alpha^2 R^3 - 2450 \alpha R^4 + 480 R^5 \right) \times \right. \]
\[
\left. \left( 4410 \alpha^5 - 7350 \alpha^3 R^2 + 8064 \alpha^2 R^3 - 3675 \alpha R^4 + 640 R^5 \right) \frac{\delta_{ij}}{5880} \right. \]
\[
\left. \frac{R_{sj} R_{ij}}{84} \left( 126 \alpha^5 - 420 \alpha^3 R^2 + 576 \alpha^2 R^3 - 315 \alpha R^4 + 64 R^5 \right) \right\} \]
\[ (44) \]
\[ \eta_{ij} = \frac{R}{1008\alpha^5} \left( -126\alpha^5 + 420\alpha^3R^2 - 576\alpha^2R^3 + 315\alpha R^4 - 64R^5 \right) \times \left( \frac{1}{(1-\nu)} \left( v (N_jR_{i,\nu} + \delta_jR_{i,N}) - 3R_{i,\nu} R_{i,N} \right) + N_jR_{i,j} \right) \]

\[ \left( 378\alpha^5 - 2100\alpha^2R^2 + 3456\alpha^2R^3 - 2205\alpha R^4 + 512R^5 \right) \times \left( \frac{1}{(1-\nu)} \left( v N_jR_{i,j} - R_{i,\nu} R_{i,j} R_{i,N} \right) + N_jR_{i,j} + \delta_jR_{i,N} \right) \] (45)

For \( R \geq \alpha \), substitute from Eq. (35) into Eqs. (25-27), then into Eqs. (4-5), it gives:

\[ \Psi_{ij} = \frac{1}{(1-\nu)\mu} \left[ \delta_{ij} \left[ \frac{\alpha^2}{56} \left( (1 + \ln R) + \frac{\alpha^2}{23520} (109 - 420 \ln \alpha) \right) \left( 7 - 8\nu \right) - \frac{\alpha^2}{23520} (109 - 420 \ln \alpha) + \frac{5\alpha^4}{2016R^2} \right] - R_{i,\nu} R_{i,j} \left[ \frac{\alpha^2}{56} - \frac{5\alpha^4}{2016R^2} \right] \right] \] (46)

\[ \eta_{ij} = \frac{\alpha^2}{28R} \left[ -1 + \frac{5\alpha^4}{36R^2} \left( \frac{1}{(1-\nu)} \left( v (N_jR_{i,\nu} + \delta_jR_{i,N}) - 3R_{i,\nu} R_{i,j} \right) + N_jR_{i,j} \right) + \frac{1}{(1-\nu)} \left( v N_jR_{i,j} - R_{i,\nu} R_{i,j} R_{i,N} \right) + N_jR_{i,j} + \delta_jR_{i,N} \right] \] (47)

4.3 The Third Function
For \( R \leq \alpha \), substitute from Eqs. (16-18) into (4-5), it gives:

\[ \Psi_{ij} = \frac{R^2}{44(1-\nu)\mu \alpha^8} \left\{ 44(1-\nu)(11340\alpha^8 - 26460\alpha^6R^2 + 88200\alpha^4R^4 - 138240\alpha^2R^6 + 99225\alpha^4R^8 - 35840\alpha^6R^7 + 5292\alpha^8) \left( 62370\alpha^6 + 97020\alpha^4R^2 + 242550\alpha^4R^4 - 337920\alpha^2R^6 + 218295\alpha^2R^8 - 71680\alpha^6R^7 + 97020\alpha^8 \right) \right\} \times \frac{\delta_{ij} - R_{i,\nu} R_{i,j}}{15120} \times \frac{1}{36} \]

\[ (297\alpha^8 - 924\alpha^6R^2 + 3465\alpha^4R^4 - 5632\alpha^2R^6 + 4158\alpha^2R^8 - 1536\alpha R^7 + 231R^8) \] (48)
\[ \eta_{ij} = \frac{R}{792\alpha^8} \left( -297\alpha^8 + 924\alpha^6 R^2 - 3465\alpha^4 R^4 + 5632\alpha^2 R^6 - 4158\alpha R^8 - 231R^{10} \right) \times \left[ \frac{1}{(1 - \nu)} \left( v (N_i R_{i,j} + \delta_{ij} R_{i,N} - 3 R_{i,j} R_{j,N}) + N_i R_{j} \right) \right] + \left[ \frac{1}{(1 - \nu)} \left( v N_i R_{i,j} - R_{i,j} R_{j,N} + N_j R_{i} + \delta_{ij} R_{i,N} \right) \right] \]

For \( R \geq \alpha \), substitute from Eq. (37) into Eqs. (25-27), then into Eqs. (4-5), it gives:

\[ \Psi_{ij} = \frac{1}{(1 - \nu)\mu} \left\{ \frac{\alpha^2}{24} \left[ \frac{\alpha^2}{24} (1 + \ln R) + \frac{\alpha^4}{60480} (997 - 2520 \ln \alpha) \right] \right\} \left( 7 - 8\nu \right) - \left[ \frac{\alpha^2}{60480} (997 - 2520 \ln \alpha) + \frac{7\alpha^4}{15848R^2} \right] - R_{i,j} R_{j,i} \left( \frac{\alpha^2}{24} - \frac{7\alpha^4}{15848R^2} \right) \]

\[ \eta_{ij} = \frac{\alpha^2}{12R} \left\{ -1 + \frac{7\alpha^2}{66R^2} \left[ \frac{1}{(1 - \nu)} \left( v (N_i R_{i,j} + \delta_{ij} R_{i,N} - 3 R_{i,j} R_{j,N}) + N_i R_{j} \right) \right] + \left[ \frac{1}{(1 - \nu)} \left( v N_i R_{i,j} - R_{i,j} R_{j,N} + N_j R_{i} + \delta_{ij} R_{i,N} \right) \right] \right\} \]

4.4 The Fourth Function

For \( R \leq \alpha \), substitute from Eqs. (20-22) into (4-5), it gives:

\[ \Psi_{ij} = \frac{R^2}{4(1 - \nu)\mu} \left\{ 4(1 - \nu) (2007720\alpha^{11} - 5521230\alpha^9 R^2 + 14723280\alpha^7 R^4 - 57972915\alpha^5 R^6 + 104698880\alpha^4 R^7 - 92756664\alpha^3 R^8 + 46725120\alpha^2 R^9 - 12882870\alpha R^{10} + 1520640 R^{11} \right) - (1003860\alpha^{11} - 1840410\alpha^9 R^2 + 3680820\alpha^7 R^4 - 11594583\alpha^5 R^6 + 19036160\alpha^4 R^7 - 15459444\alpha^3 R^8 + 7188480\alpha^2 R^9 - 1840410\alpha R^{10} + 202752 R^{11}) \right\} \]

\[ \frac{\delta_{ij}}{8030880} \left( \frac{R_{i,j}}{780} \right) (195\alpha^{11} - 715\alpha^9 R^2 + 2145\alpha^7 R^4 - 9009\alpha^5 R^6 + 16640\alpha^4 R^7 - 15015\alpha^3 R^8 + 7680\alpha^2 R^9 - 2145\alpha R^{10} + 256 R^{11}) \]

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Web: www.witpress.com Email witpress@witpress.com
Paper from: Boundary Elements XXIV, CA Brebbia, A. Tadeu and V. Popov (Editors).
ISBN 1-85312-914-3
\[ \eta_{ij} = \frac{R}{1560 \alpha^2} \left[ -195 \alpha^{11} + 715 \alpha^9 \gamma^2 - 2145 \alpha^7 \gamma^4 + 9009 \alpha^5 \gamma^6 - 16640 \alpha^4 \gamma^7 + 15015 \alpha^3 \gamma^8 - \\ 7680 \alpha^2 \gamma^9 + 2145 \alpha \gamma^{10} - 256 \gamma^{11} \right] \left[ \frac{1}{(1-v)} \left( \nu (N_j R_{r_i} + \delta_{ij} R_{r_N}) - 3 R_{r_i} R_{r_j} R_{r_N} \right) + N_j R_{r_j} \right] + \\ \left( 585 \alpha^{11} - 3575 \alpha^9 \gamma^2 + 15015 \alpha^7 \gamma^4 - 81081 \alpha^5 \gamma^6 + 16640 \alpha^4 \gamma^7 - 165165 \alpha^3 \gamma^8 + \\ 92160 \alpha^2 \gamma^9 - 27885 \alpha \gamma^{10} + 3584 \gamma^{11} \right) \left[ \frac{1}{(1-v)} \left( \nu (N_j R_{r_j} - R_{r_i} R_{r_j} R_{r_N}) + N_j R_{r_i} + \delta_{ij} R_{r_N} \right) \right] \right] \tag{53} \]

For \( R \geq \alpha \), substitute from Eq. (39) into Eqs. (25-27), then into Eqs. (4-5), it gives:

\[ \Psi_{ij} = \frac{1}{(1-v) \mu} \left\{ \frac{\delta_{ij}}{2} \left[ \left( \frac{7 \alpha^2}{624} (1 + \ln R) + \frac{7 \alpha^2}{32123520} (25943 - 51480 \ln \alpha) \right) (7 - 8v) - \\ \left( \frac{7 \alpha^2}{32123520} (25943 - 51480 \ln \alpha) + \frac{\alpha^4}{1040 R^2} \right) \right] - R_{r_i} R_{r_j} \left[ \frac{7 \alpha^2}{624} - \frac{\alpha^4}{1040 R^2} \right] \right\} \tag{54} \]

\[ \eta_{ij} = \frac{\alpha^2}{104 R} \left\{ - \frac{7}{3} + \frac{\alpha^2}{5 R^2} \left[ \frac{1}{(1-v)} \left( \nu (N_j R_{r_i} + \delta_{ij} R_{r_N}) - 3 R_{r_i} R_{r_j} R_{r_N} \right) + N_j R_{r_j} \right] + \\ \left( \frac{7}{3} + \frac{\alpha^2}{5 R^2} \right) \left[ \frac{1}{(1-v)} \left( \nu (N_j R_{r_j} - R_{r_i} R_{r_j} R_{r_N}) + N_j R_{r_i} + \delta_{ij} R_{r_N} \right) \right] \right\} \tag{55} \]

### 3 Summary and Conclusions

The CS RBF has been successfully used in the implementation of the DRM for Elasticity problems. The mathematical formulation was set up for four functions. All relevant kernels were derived and given in explicit form.

### 4 References


