The dual reciprocity method for solving biharmonic problems

A J Davies, W Toutip and S J Kane

Mathematics Department, University of Hertfordshire, UK.
Department of Mathematics, Khon Kaen University, Thailand.

Abstract

The dual reciprocity method is now established as a suitable approach to the boundary element method solution of non-homogeneous field problems. The Poisson problem was probably the first such problem to be solved using dual reciprocity and has been the subject of much interest. By introducing a secondary dependent variable biharmonic problems may be written as a pair of coupled Poisson-type problems and as such are amenable to a dual reciprocity approach. The procedure is straightforward but some care is required when applying boundary conditions. If the boundary conditions can be expressed explicitly in terms of the primary variable and the secondary variable then the equations uncouple. If however, the boundary conditions are expressed in terms of the primary variable only then a fully coupled system must be solved. The process is well-suited to the analysis of the bending of a flat plate. Simply-supported and clamped boundary conditions correspond respectively to the two cases.

1 The dual reciprocity method for Poisson problems

We shall first give a brief overview of the dual reciprocity method for Poisson problems [1] governed by the partial differential equation

\[ \nabla^2 u = b(x, y) \]  

in \( \Omega \) subject to suitable boundary conditions on \( \Gamma \).

The boundary-value problem is replaced by an equivalent integral equation defined only on the boundary, \( \Gamma \). The method is then directly analogous to the boundary element method in that the boundary is discretised using \( N \) nodes so that the curve, \( \Gamma \), is approximated by a piecewise curve \( \Gamma_N \), enclosing an ap-
proximation, $\Omega_N$, to $\Omega$. The integral formulation is then replaced by an approximating system of algebraic equations. The non-homogeneous term in equation (1.1) is “taken to the boundary” by approximating $b(x, y)$ by a linear combination of functions, $f_j(x, y)$, each of which yields particular solutions to Laplace’s equation. The second form of Green’s theorem then provides a reciprocal relation from which domain integrals become boundary integrals.

In order to approximate $b(x, y)$ we choose another $L$ nodes in the interior, $\Omega_N$, and write

$$ b \bigg[ \sum_{j=1}^{N+L} \alpha_j f_j(x, y) \bigg], $$

(1.2)

where

$$ \nabla^2 \hat{u}_j = f_j(x, y) $$

(1.3)

for some $\hat{u}_j$.

We shall use thin plate splines $r_i^2 \ln r_i$, augmented with a linear term of the form $a+bx+cy$ for the $f_i$.

The corresponding particular solutions are

$$ \hat{u}_j = \frac{1}{4} r_j^2 \ln r_j - \frac{1}{32} r_j^4, $$

$$ \hat{q}_j = \left( \frac{1}{4} r_j^2 \ln r_j - \frac{1}{16} r_j^4 \right) \left( (x-x_j) \partial x_j / \partial n + (y-y_j) \partial y_j / \partial n \right), $$

$$ \hat{u}_{N+L+1} = \frac{1}{4} (x^2 + y^2), \quad \hat{u}_{N+L+4} = \frac{1}{2} (x \partial x / \partial n + y \partial y / \partial n), $$

$$ \hat{u}_{N+L+2} = \frac{1}{6} x^3, \quad \hat{u}_{N+L+5} = \frac{1}{2} x^2 \partial x / \partial n, $$

$$ \hat{u}_{N+L+3} = \frac{1}{6} y^3, \quad \hat{u}_{N+L+6} = \frac{1}{2} y^2 \partial y / \partial n. $$

By collocating at the $N+L$ nodes, the approximation (1.2) leads to a system of equations of the form

$$ \mathbf{b} = \mathbf{F} \mathbf{\alpha}. $$

(1.4)

The integral equation equivalent to equation (1.1) is [1]

$$ c_i u_i + \int_{\Gamma} q^* u d\Gamma - \int_{\Omega} u^* q d\Omega = \int_{\Omega} b u^* d\Omega. $$

(1.5)

Using equations (1.2) and (1.3) and applying Green’s theorem, the boundary element approximation to equation (1.5) may be written in the form

$$ c_i u_i + \sum_{k=1}^{N} \int_{\Gamma_k} q^* u d\Gamma - \sum_{k=1}^{N} \int_{\Gamma_k} u^* q d\Gamma = $$

$$ \sum_{j=1}^{N+L} \alpha_j \left( c_j \hat{u}_j + \sum_{k=1}^{N} \int_{\Gamma_k} \hat{q}_j u d\Gamma - \sum_{k=1}^{N} \int_{\Gamma_k} u^* \hat{q}_j d\Gamma \right), $$

(1.6)

for $i = 1...N$.

In the following derivation we use subscripts $I$ and $B$ to denote values associated with internal and boundary points respectively so that we may write equation (1.6) in matrix form as
Internal values are given by
\[
U_i = -\sum_{k=1}^{N} \int_{\Gamma_k} q^*u d\Gamma + \sum_{k=1}^{N} \int_{\Gamma_k} u^*q d\Gamma + 
\sum_{j=1}^{N} \alpha_j \left( c_j \mu_j + \sum_{k=1}^{N} \int_{\Gamma_k} q^* \hat{u}_j d\Gamma - \sum_{k=1}^{N} \int_{\Gamma_k} \dot{q}^* d\Gamma \right),
\]
which we may write in matrix form as
\[
\begin{bmatrix} U_{I} \end{bmatrix} = \begin{bmatrix} H & B \end{bmatrix} \begin{bmatrix} Q_{I} \end{bmatrix} - \begin{bmatrix} G \end{bmatrix} \begin{bmatrix} U_{I} \end{bmatrix} + \begin{bmatrix} \alpha \end{bmatrix} + IU\begin{bmatrix} \alpha \end{bmatrix}.
\]
(1.9)

Equations (1.7) and (1.9) may be combined, using equation (1.4), in the form
\[
HU - GQ = \begin{bmatrix} H & \alpha \end{bmatrix} F^{-1} \begin{bmatrix} \alpha \end{bmatrix} - \begin{bmatrix} H & B \end{bmatrix} \begin{bmatrix} Q_{I} \end{bmatrix} F^{-1} \begin{bmatrix} \alpha \end{bmatrix} - \begin{bmatrix} H & B \end{bmatrix} \begin{bmatrix} U_{I} \end{bmatrix} + \begin{bmatrix} \alpha \end{bmatrix} + IU\begin{bmatrix} \alpha \end{bmatrix}.
\]
(1.10)

from which we find the boundary solution, \( U_{I} \) and \( Q_{I} \), and the internal solution, \( U_{I} \), simultaneously.

The matrices in equation (1.10) have the following forms:
\[
H = \begin{bmatrix} H_{BB} & 0 \\
B_{IB} & 1 \end{bmatrix}, 
G = \begin{bmatrix} G_{BB} & 0 \\
G_{IB} & 0 \end{bmatrix}, 
U = \begin{bmatrix} U_{B} \\
U_{I} \end{bmatrix}, 
Q = \begin{bmatrix} Q_{B} \\
0 \end{bmatrix}.
\]
(1.11)

We define the matrix \( S \), which depends only on the geometry, by
\[
S = \begin{bmatrix} H & G \end{bmatrix} F^{-1}.
\]
(1.12)
then equation (1.10) is written
\[
HU - GQ = Sb.
\]
(1.13)

2 Biharmonic problems

We consider the biharmonic problem
\[
\nabla^4 u = \nabla^2 \left( \nabla^2 u \right) = b(x, y)
\]
in \( \Omega \).

We shall use the terminology primary variable for the unknown function \( u \) and we introduce the secondary variable, \( v \), given by
\[
v \equiv \nabla^2 u.
\]
(2.2)

Using equation (2.2) in equation (2.1) we obtain the pair of coupled Poisson problems
\[
\nabla^2 u = v,
\]
(2.3)
\[
\nabla^2 u = b(x, y).
\]
(2.4)

We seek the solution of equation (2.1) subject to boundary conditions
\[
u = g_1(x, y)
\]
(2.5)
and
\[
either \frac{\partial u}{\partial n} = g_2(x, y)
\]
(2.6)
or
\[
v = g_3(x, y)
\]
(2.7)
at each point \((x, y)\) on \(\Gamma\).

Now the dual reciprocity method [3] may be applied to each of the equations (2.3) and (2.4) to obtain systems of equations of the form

\[
\begin{align*}
HU - GQ &= SV, \\
HV - GP &= Sb.
\end{align*}
\]

(2.8) (2.9)

where \(U, Q, V\) and \(P\) are vectors of the unknowns \(u, q, v\) and \(p \equiv \frac{\partial v}{\partial n}\).

Now equations (2.8) and (2.9) are weakly coupled. In principle we could solve equation (2.9) for \(V\) then substitute in equation (2.8) to find \(U\). However, this is possibly only if we know both \(u\) and \(v\) on the boundary. In general this is not the case. We can see this very clearly if we combine equations (2.8) and (2.9) as a single equation as follows:

\[
\begin{bmatrix}
I & -G_{IB}^2 & -S_{IB}^1 & 0 \\
0 & -G_{BB}^2 & -S_{BB}^1 & 0 \\
0 & 0 & I & -G_{IB}^1 \\
0 & 0 & 0 & H_{BB}^1
\end{bmatrix}
\begin{bmatrix}
U \\
Q_B \\
V_I \\
V_B^1 \\
P
\end{bmatrix}
= 
\begin{bmatrix}
U_B \\
Q_B^2 \\
V_B \\
b
\end{bmatrix}
\]

(2.10)

Superscripts 1 and 2 are used to denote the form of boundary condition which applies. We assume that there are \(M\) points at which boundary condition (2.6) applies and denote such points with the superscript 1. Similarly we have \(N - M\) points at which boundary condition (2.7) applies and denote such points with the superscript 2. We can now see that if we have no boundary conditions of the form (2.6), \(i.e.\ M = 0\), then the equations uncouple and we solve a pair of \((N+L)\times(N+L)\) equations rather than a \((2N+2L)\times(2N\times2L)\) single, much larger, system.

Toutip et al. [2] have shown that, for Poisson problems, boundary derivatives are computed more accurately if the gradient method [4] is used at points of discontinuity at node \(k\). Here, \(u(x, y)\) is interpolated linearly over the triangle defined by the consecutive nodes \(k-1, k, k+1\) in terms of the nodal values \(u_{k-1}, u_k, u_{k+1}\). This linear approximation then yields \(\text{grad}\ u\) over the triangle from which we find \(q = \text{grad}\ u \cdot \mathbf{n}\) on each element at node \(k\). At such points a single unknown variable, \(\left|\nabla u\right|_k\), occurs in the overall system of equations from which the flux in each element may be obtained. Now for biharmonic problems, the boundary condition (2.5) means that we can always use the gradient method
in the computation of \( q \). Similarly, if we have boundary condition (2.7) we can use the gradient method for \( p \). However, if we have boundary condition (2.6) then we are not able to use the method for \( p \).

3 Computational results

We consider three examples defined on the unit square.

3.1 Non-homogeneous problem

Consider the biharmonic problem

\[
\nabla^4 u = 48
\]

in the unit square \( \{(x, y): 0 < x < 1, 0 < y < 1\} \) with boundary conditions on \( u \) and \( q \) consistent with the exact solution \( u = x^4 + y^4 \).

We consider two discretisations, each with linear boundary elements: \( N=16, L=9 \) and \( N=32, L=49 \). The internal values of \( u \) are shown in Table 3.1 from which we see convergence to the exact solution. In this example we have used the gradient approach in the computation of \( u \). However, the boundary conditions do not allow the gradient method to be used in the computation of \( v \). Nevertheless, we see from Table 3.2 that the values of \( v \) along the line \( y = 0.5 \) agree well with the exact values.

Table 3.1. Internal values of \( u \), \( N=16, L=9 \) and \( N=32, L=49 \).

<table>
<thead>
<tr>
<th>Point</th>
<th>16, 9</th>
<th>32, 49</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.25, 0.25)</td>
<td>0.0098</td>
<td>0.0083</td>
<td>0.0078</td>
</tr>
<tr>
<td>(0.5, 0.25)</td>
<td>0.0695</td>
<td>0.0672</td>
<td>0.0664</td>
</tr>
<tr>
<td>(0.75, 0.25)</td>
<td>0.3216</td>
<td>0.3208</td>
<td>0.3203</td>
</tr>
<tr>
<td>(0.25, 0.5 )</td>
<td>0.0695</td>
<td>0.0672</td>
<td>0.0664</td>
</tr>
<tr>
<td>(0.5, 0.5 )</td>
<td>0.1288</td>
<td>0.1260</td>
<td>0.1250</td>
</tr>
<tr>
<td>(0.75, 0.5 )</td>
<td>0.3801</td>
<td>0.3795</td>
<td>0.3789</td>
</tr>
<tr>
<td>(0.25, 0.75)</td>
<td>0.3216</td>
<td>0.3208</td>
<td>0.3203</td>
</tr>
<tr>
<td>(0.5, 0.75)</td>
<td>0.3801</td>
<td>0.3795</td>
<td>0.3789</td>
</tr>
<tr>
<td>(0.75, 0.75)</td>
<td>0.6333</td>
<td>0.6333</td>
<td>0.6328</td>
</tr>
</tbody>
</table>
3.2 Plate bending

The deflection, \( w \), of a flat plate due to a lateral load density, \( \rho \), is given by [5]

\[
\nabla^4 w = \frac{\rho}{D},
\]

where \( D \) is the flexural rigidity of the plate.

By setting \( u = \frac{\rho w}{\rho} \) equation (3.1) becomes

\[
\nabla^4 u = 1.
\]

For a simply-supported plate with straight sides the boundary conditions, viz zero displacement and zero bending moment, are given in terms of both the primary and secondary variables [6]:

\[
u = 0 \quad \text{and} \quad v = \nabla^2 u = 0.\]

For a clamped plate the boundary conditions, viz zero displacement and zero slope, are given in terms of the primary variable only:

\[
u = 0 \quad \text{and} \quad q = 0.\]

In the following two examples we consider the flat plate bounded by the unit square \( \{(x, y) : 0 < x < 1, 0 < y < 1\} \). We shall compute the maximum deflection, \( u_{\text{max}} \), at the centre of the plate in each case and compare with a finite element solution [7] and the exact value [5]. The finite element solution is given using linear triangles on meshes with 16, 64, 144 and 256 nodes. We shall use boundary discretisations with 8, 16, 32, 48 and 64 linear elements with 1, 9, 49, 121 and 225 internal nodes respectively.

Simply-supported plate

In this case we have the boundary condition (3.3) and we may use the gradient method for both the primary and the secondary variables. We see from Table 3.3 that the dual reciprocity method compares very well with the finite element method with a similar number of unknowns.

<table>
<thead>
<tr>
<th>( x )</th>
<th>16, 9</th>
<th>32, 49</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>4.65</td>
<td>4.56</td>
<td>4.563</td>
</tr>
<tr>
<td>0.25</td>
<td>4.52</td>
<td>4.54</td>
<td>4.547</td>
</tr>
<tr>
<td>0.5</td>
<td>4.51</td>
<td>4.50</td>
<td>4.500</td>
</tr>
<tr>
<td>0.75</td>
<td>4.39</td>
<td>4.41</td>
<td>4.422</td>
</tr>
<tr>
<td>1.0</td>
<td>4.40</td>
<td>4.31</td>
<td>4.313</td>
</tr>
</tbody>
</table>

Table 3.2. Values of \( v \) along \( y = 0.5 \), \( N = 16, L = 9 \) and \( N = 32, L = 49 \).
Table 3.3. Maximum deflection, exact value is 0.004062.

<table>
<thead>
<tr>
<th>Dual reciprocity</th>
<th>Finite elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>8, 1</td>
<td>16, 0.003036</td>
</tr>
<tr>
<td>9, 16</td>
<td>64, 0.003939</td>
</tr>
<tr>
<td>32, 49</td>
<td>144, 0.004033</td>
</tr>
</tbody>
</table>

Clamped plate
In this case we have the boundary condition (3.4) and we cannot use the gradient approach for the secondary variable. We see from Table 3.4 that convergence is slower than for the simply-supported plate and that again the dual reciprocity method compares well with the finite element.

Table 3.4. Maximum deflection, exact value is 0.00126.

<table>
<thead>
<tr>
<th>Dual reciprocity</th>
<th>Finite elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>16, 9</td>
<td>16, 0.001403</td>
</tr>
<tr>
<td>32, 49</td>
<td>64, 0.001304</td>
</tr>
<tr>
<td>48, 121</td>
<td>144, 0.001283</td>
</tr>
<tr>
<td>64, 225</td>
<td>256, 0.001275</td>
</tr>
</tbody>
</table>

4 Conclusions

The dual reciprocity method provides a suitable process for solving problems defined by the biharmonic operator. When the partial differential equation is written as a pair of coupled Poisson problems, the gradient method can always be applied to improve boundary flux computations for the primary variable. However, it can only be used for the secondary variable in the case of a Dirichlet-type boundary condition. In the former case the pair of system equations uncouples and may be solved sequentially. However, in the latter case the equations do not uncouple and the pair of equations must be solved simultaneously.

The ideas developed above can be applied to any problem which may be reduced to a pair of Poisson-type problems. e.g. Toutip [3] and Davies et al. [8] have considered non-linear problems in the modelling of ohmic heating of food materials.

References


