The Laplace transform dual reciprocity boundary element method for nonlinear transient field problems

M.E.Honnor and A.J.Davies

Department of Mathematics, University of Hertfordshire, UK.

Abstract

The Laplace transform method in the time domain provides an alternative to time marching schemes for nonlinear transient field problems. The advantages over time marching schemes are that there are no timestep limits imposed due to accuracy and stability. Furthermore since the solution at a particular time is not dependant on the solution at any other time, apart from initial conditions, the method is suitable for the solution in a distributed memory environment, with virtually no communication overhead between processors, as each processor is working on the solution at a different time. The Laplace Transform Dual Reciprocity Boundary Element Method has been implemented on a 512 processor nCUBE machine arranged in a hypercube configuration.

1 Introduction

A number of physical processes are governed by the so called quasi-harmonic equation including heat conduction, gas diffusion, seepage and compressible flow, magneto-statics, torsion and Reynolds film lubrication. These processes are generally termed field problems. Transient field problems can be solved using the boundary element method in one of two ways, the time domain approach and the transform approach. The time domain approach uses a time marching scheme whereby the solution is sought at each timestep before progressing to the next timestep. The time marching scheme will generally have timestep limits imposed due to accuracy and stability and can be numerically inefficient. The transform approach on the other hand employs a transform where the time dependant derivative is replaced by a transform variable. Once the solution is obtained in
transform space at a particular time, reconstituting the solution requires an inverse transform. The accuracy obtained depends upon this inverse transform. For field problems the Laplace transform is usually employed, the inverse transform is then performed numerically using the method of Stehfest [1,2]. The Dual Reciprocity Method is a technique for taking domain integrals to the boundary in the Boundary Element Method [3]. The Laplace Transform Dual Reciprocity Boundary Element Method, LTDRM, has been studied for linear and nonlinear transient field problems [4,5,6,7,8,9].

2 Heat Conduction

The heat conduction equation for two-dimensional problems for isotropic materials is

$$\frac{\partial}{\partial x}\left(K \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial u}{\partial y}\right) + V = \rho c \frac{\partial u}{\partial t}. \tag{1}$$

Subject to boundary conditions

- Dirichlet boundary condition, prescribed temperature
  $$u = u. \tag{2}$$

- Neumann boundary condition, prescribed flux
  $$q_f = q_f = K \frac{\partial u}{\partial n}. \tag{3}$$

- Convection boundary condition
  $$q_c = h(u_c - u). \tag{4}$$

- Radiation boundary condition
  $$q_R = \sigma \varepsilon (u_R^4 - u^4). \tag{5}$$

Where $u$ is the temperature, $K$ is the thermal conductivity, $V$ is the heat generated, $\rho$ is the density, $c$ is the specific heat, $h$ is the convection transfer coefficient, $u_c$ is the ambient temperature for convection, $\sigma$ is the Stefan-Boltzmann constant $= 5.667 \times 10^8$, $\varepsilon$ is the surface emissivity and $u_R$ is the ambient temperature for radiation. From equation (1)
In order to proceed it is now necessary to define how the thermal conductivity $K$, is dependent upon temperature $u$, following [7]

$$K = K_0 (1 + \beta_K u).$$

This type of dependency is the same as that suggested by [10], where for most substances $\beta_K$ is small and negative. Substituting into equation (6) gives

$$\nabla^2 u = \frac{\rho c}{K} \frac{\partial u}{\partial t} - V - \frac{\beta_K}{K} \left( \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} \right).$$

### 3 The Dual Reciprocity Method

The Laplace operator is isolated on the left hand side and all other terms are transferred to the right hand side to form an equation of the type

$$\nabla^2 u = b(x, y, u).$$

In order to take the right hand side $b(x, y, u)$ to the boundary the approximation of $b$ is written as

$$b_i = \sum_{j=1}^{N+L} f_y^i \alpha_j,$$

where $b_i$ is the function $b$ at node $i$, $f_y^i$ are approximating functions and $\alpha_i$ unknown coefficients. The approximation is done at $(N+L)$ nodes called DRM collocation points, $N$ boundary nodes and $L$ internal nodes. The functions $f$ are defined by

$$\nabla^2 \hat{u} = f,$$

where $\hat{u}$ is a particular solution. Combining equations (9), (10) and (11) gives

$$\nabla^2 u = -\sum_{j=1}^{N+L} \left( \nabla^2 \hat{u}_j \right) \alpha_j.$$

Multiplying by the fundamental solution $u^*$ and integrating by parts gives [3]
where \( q = \partial u / \partial n \), after discretization this becomes

\[
c_i u_i + \sum_{k=1}^{N} H_{ik} u_k - \sum_{k=1}^{N} G_{ik} q_k = \sum_{j=1}^{N+L} \alpha_j \left( c_i \hat{u}_j + \sum_{k=1}^{N} H_{ik} \hat{u}_k - \sum_{k=1}^{N} G_{ik} \hat{q}_k \right),
\]

which is written for each of the \((N + L)\) nodes \(i\), and incorporating the \( c_i \) terms into the diagonal of \( H \) gives

\[
Hu - Gq = \left( H \hat{U} - G \hat{Q} \right) \alpha.
\]

From equation (10), \( b = F \alpha \), hence

\[
\alpha = F^{-1} b,
\]

which is substituted into equation (15) to give

\[
Hu - Gq = Sb,
\]

where \( S = \left( H \hat{U} - G \hat{Q} \right) F^{-1} \).

The matrices \( \hat{U} \), \( \hat{Q} \) and \( F \) are all known if \( f \) is defined. Two frequently used approximating functions are [11] the linear function \( f = 1 + r \) and the augmented thin plate spline \( f = r^2 \log r + ax + by + c \).

### 4 The Laplace Transform Dual Reciprocity Boundary Element Method

Before we can take the Laplace transform of equation (8) it has to be linearised. Rewriting equation (8) in a iterative format where the solution is sought at a specific time gives

\[
\nabla^2 u = \frac{\rho c}{K} \partial u \partial t - \frac{\hat{V} u}{K} - \frac{\beta_k}{(1 + \beta_k) u} \left( \partial \hat{u} \partial x \partial x + \partial \hat{u} \partial y \partial y \right),
\]

where \( \sim \) over a quantity implies the value at the previous iteration and if \( V \) is a function of \( u \), \( V' = V / u \). Taking the Laplace Transform of equation (18) gives
spatial coordinate, say $x$ can be written as
\[ U. \]

The Dual Reciprocity Method approximation to a derivative with respect to $x$ where $h$ is the Laplace parameter and $U$ is the Laplace Transform variable of $u$. The Dual Reciprocity Method approximation to a derivative with respect to a spatial coordinate, say $x$ can be written as [3]
\[
\frac{\partial u}{\partial x} = \frac{\partial F}{\partial x} F^{-1} u.
\] (20)

Defining a diagonal matrix $T_x$ such that
\[
T_x(i,i) = \frac{\beta_k}{1 + \beta_k u} \frac{\partial u_i}{\partial x} = \frac{\beta_k}{1 + \beta_k u} \frac{\partial F}{\partial x} F^{-1} u.
\] (21)

Similarly for $T_y$ enables the last term in equation (19) to be written as
\[
\frac{\beta_k}{(1 + \beta_k u)} \left( \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{u}}{\partial y} \right) \left( T_x \frac{\partial F}{\partial x} + T_y \frac{\partial F}{\partial y} \right) F^{-1} U = \tilde{C} U.
\] (22)

Defining another diagonal matrix $B$ such that
\[
B(i,i) = V_{i,i}.
\] (23)

Applying the Dual Reciprocity Method, equation (19) gives
\[
H U - G Q = S \left\{ \tilde{\rho} c \left( \lambda U - u_0 \right) - \frac{\tilde{B}}{K} U - \tilde{C} U \right\}.
\] (24)

Rearranging gives the final matrix equation
\[
\left\{ H + S \left( -\lambda \frac{\tilde{\rho} c}{K} + \frac{\tilde{B}}{K} + \tilde{C} \right) \right\} U - G Q = -S \frac{\tilde{\rho} c}{K} u_0.
\] (25)

## 5 Boundary Conditions

Before solving the system of equations (25) it is first necessary to examine how to apply the boundary conditions in Laplace Transform space.
Dirichlet boundary condition, prescribed temperature

\[ U = \frac{\bar{u}}{\lambda}. \]  

(26)

Neumann boundary condition, prescribed flux

\[ Q = \frac{\partial U}{\partial n} = \frac{-q_f}{\lambda K}. \]  

(27)

Convection boundary condition

\[ Q = \frac{\partial U}{\partial n} = \frac{h}{K} \left( \frac{u_c}{\lambda} - U \right). \]  

(28)

Radiation boundary condition

The radiation boundary condition is first linearised

\[ q_R = \sigma e \left( u_R^4 - u^4 \right) = \sigma e \left( u_R^2 + u^2 \right) \left( u_R^2 - u^2 \right) = \sigma e \left( u_R^2 + u^2 \right) (u_R + u)(u_R - u). \]  

(29)

Before taking the Laplace transform to give

\[ Q = \frac{\partial U}{\partial n} = \frac{\sigma e}{K} \left( \frac{u_R^2 + u^2}{\lambda} \right) \left( u_R + u \right) \left( \frac{u_R}{\lambda} - U \right). \]  

(30)

6 Numerical Examples

The problems associated with corners and discontinuous boundary conditions have been handled via the gradient approach [12]. In all examples presented, Stehfest’s [1,2] algorithm, with \( m = 6 \), was used for the numerical Laplace inversion. Unless stated otherwise linear approximating functions are used in the dual reciprocity method. Convergence was achieved when the ratio between the norm of the iterative change of the variable, \( du \), and the norm of the variable, \( u \), was less than a small number \( e = 10^{-10} \).

\[ \frac{\sqrt{du_i du_i}}{\sqrt{u_i u_i}} \leq e. \]  

(31)

6.1 Glass square

This problem is linear and studies a uniform glass square, 2 x 2, with uniform initial temperature of zero. The boundary is suddenly heated to a temperature of unity at \( t = 0 \) sec and maintained at this temperature. The material properties are,
thermal conductivity \( K = 1.0 \) and heat capacity \( \rho c = 1.0 \). The boundary was divided into 64 elements with 49 equally spaced internal points. Table 1 shows the temperature at the centre of the square for the analytical solution [10], linear function and augmented thin plate spline results, also shown is the results from the Laplace transform Bessel function approach to the modified Helmholtz equation [13] using the gradient approach and 64 elements on the boundary only. As can be seen from the table the Laplace transform dual reciprocity method is comparable in accuracy to the Laplace transform Bessel function approach.

Table 1: Temperature at centre of glass square

<table>
<thead>
<tr>
<th>Time Sec.</th>
<th>Analytical Temp</th>
<th>( T = 1 + r )</th>
<th>( r^2 \log(r) + ax + by + c )</th>
<th>Bessel Function</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Temp</td>
<td>Error %</td>
<td>Temp</td>
<td>Error %</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1052</td>
<td>0.0981</td>
<td>6.78</td>
<td>0.1064</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4036</td>
<td>0.4127</td>
<td>2.27</td>
<td>0.4150</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6318</td>
<td>0.6340</td>
<td>0.35</td>
<td>0.6335</td>
</tr>
<tr>
<td>0.4</td>
<td>0.7749</td>
<td>0.7699</td>
<td>0.65</td>
<td>0.7687</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8625</td>
<td>0.8531</td>
<td>1.09</td>
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</tr>
<tr>
<td>0.6</td>
<td>0.9161</td>
<td>0.9050</td>
<td>1.21</td>
<td>0.9039</td>
</tr>
<tr>
<td>0.7</td>
<td>0.9488</td>
<td>0.9381</td>
<td>1.13</td>
<td>0.9371</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9687</td>
<td>0.9595</td>
<td>0.95</td>
<td>0.9588</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9809</td>
<td>0.9737</td>
<td>0.74</td>
<td>0.9731</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9883</td>
<td>0.9831</td>
<td>0.53</td>
<td>0.9827</td>
</tr>
</tbody>
</table>

6.2 Slab with radiation

A slab 0.1m in length, 0.01m thick initially at 1000K, with constant temperature of 1000K at edge AB is suddenly subjected to radiation to the ambient temperature of 300K at the other end CD as shown in Figure 1. The upper and lower surfaces are perfectly insulated. The material properties are, thermal conductivity \( K = 55.6 \text{W/m}^\circ\text{C} \), density \( \rho = 7850.0 \text{kg/m}^3 \), specific heat \( c = 460.0 \text{J/kg}^\circ\text{C} \) and the surface emissivity at end CD is \( \varepsilon = 0.98 \). The boundary was divided into 44 elements with 19 equally spaced internal points.

![Figure 1: Slab with radiation](image)

The temperature at end CD is shown in figure 2, also shown are the finite element results obtained from ELFEN, available from Rockfield Software, UK. The steady state solution can be obtained by requesting the results at a large
time, the solution obtained at end CD at 10 000 sec is 927.009K which compares well with the reference target temperature of 927K [14].

![Graph showing temperature at end CD](image)

**Figure 2: Temperature at end CD**

### 6.3 Square plate

This problem is a static problem, which can be analysed by requesting the results at a large time, with nonlinear thermal conductivity which has previously been analysed by Partridge [3]. A square, of unit side length, insulated on two sides, prescribed temperature along one side and subject to a convection boundary condition along the other side as shown in Figure 3. For the convective boundary condition the convection transfer coefficient, $h=10$, and ambient temperature, $u_e=500$. The thermal conductivity is $K=K_0(1+\beta_k u)$, where $K_0=1$ and $\beta_k=0.3$. The temperature around the boundary is shown in Figure 4 also shown are the DRM1 results from Partridge [3].

![Square plate boundary conditions](image)

**Figure 3: Square plate boundary conditions**
7 Conclusions

The Laplace transform dual reciprocity boundary element method is a powerful combination of the Laplace transform and the dual reciprocity method for the solution of linear and nonlinear transient field problems. It is particularly suited to parallel processing in a distributed memory environment since the solution at each specific time value is independent of any other. The numerical examples presented show that accurate results are obtained for problems with nonlinear boundary conditions from radiation boundary conditions and for problems with nonlinear material properties.

References


[14] NAFEMS Benchmark (BMTTA(S) - Test No. T2)