A boundary element method using DRM for nonlinear heat conduction problems

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Abstract
This paper presents a dual reciprocity boundary element method (DRBEM) for solving the steady-state heat conduction problem of temperature-dependent materials. The integral equation formulation uses the fundamental solution of the Laplace equation for homogeneous materials, and a domain integral arises in the boundary integral equation. This domain integral is transformed into boundary integrals through the dual reciprocity method by introducing a new set of radial basis functions. An iterative solution procedure is used, because material constants are temperature-dependent and hence the problem is nonlinear. The details of the proposed DRBEM are presented, and based on this formulation a computer code is developed for two-dimensional problems. Through comparison of the results obtained by the computer code with the exact ones for some examples, the usefulness of the proposed DRBEM is demonstrated.

1 Introduction

This paper is concerned with a dual reciprocity boundary element method (DRBEM)[1][2] for solving the steady-state heat conduction problem of temperature-dependent materials[3]. Most of heat conduction problems are analyzed under the assumption that the material properties are constant. Under higher temperatures, however, the materials should be considered as temperature-dependent for a more accurate modeling. In such cases, mathematical formulation and its numerical implementation are not easy due to this nonlinearity.

When the boundary element method is applied to solving the heat conduction
problem of temperature-dependent materials, it is very difficult or almost impossible to find the fundamental solution for such nonlinear problems. Therefore, the fundamental solution for the problem of homogeneous materials is used for the integral equation formulation of the nonlinear problem under consideration. The well-known fundamental solution of the Laplace equation can be employed for this purpose, but a domain integral arises in the boundary integral equation[4]. As has been well known, the Kirchhoff transformation is applicable to the solution of the problem[5]. However, the pseudo-linear formulation resulting from the Kirchhoff transformation is very complicated and its numerical implementation is troublesome, in particular, when a general type of the boundary condition and/or interface of the domain should be taken into account.

The domain integral is transformed into boundary integrals through the dual reciprocity introducing a new set of radial basis functions[6][7][8]. An iterative solution procedure is then used for the solution of the nonlinear system of equations. The details of the proposed DRBEM are presented, and a computer code is developed for two-dimensional problems. The present radial basis function was successfully applied in authors’ previous paper[9] to the linear problem in homogeneous materials. This paper reports on the dual reciprocity formulation and its solution for the nonlinear problem of temperature-dependent materials. Through comparison of the results obtained by the computer code with the exact ones for a couple of problems, the usefulness of the proposed DRBEM is demonstrated.

2 Theory

2.1 Differential equation

The present paper is concerned with the steady-state heat conduction problem in which the thermal conductivity is a function which depends on temperature. The governing differential equation can be expressed by

$$\nabla \{ \lambda(u) \nabla u(x) \} = 0$$

where \( u \) is the temperature, \( \lambda \) the thermal conductivity coefficient depending on temperature, \( \nabla \) the gradient operator. The boundary conditions are expressed as follows:

$$u(x) = \bar{u}, \quad x \in \Gamma_u$$

$$\frac{\partial u(x)}{\partial n} = \bar{q}, \quad x \in \Gamma_q$$

where \( n \) is the outward normal to the boundary \( \Gamma \), and \( \partial()/\partial n \) the normal derivative. The bar denotes a prescribed value.
The governing differential equation is then expressed by

$$\nabla^2 u(x) = -\frac{1}{\lambda(u)} \nabla \lambda(u) \nabla u(x)$$  \hspace{1cm} (3)

2.2 Integral equation

In this study, the fundamental solution of Laplace equation is used for the integral equation formulation. This fundamental solution is expressed for two-dimensional problems by

$$u^*(x, y) = \frac{1}{2\pi} \ln \left( \frac{1}{r} \right)$$  \hspace{1cm} (4)

where $r$ is the distance between source point $y$ and field point $x$. If we use this fundamental solution as a weighting function, we can get from Eq. (3) the following weighted residual expression:

$$\int_{\Omega} u^*(x, y) \nabla^2 u(x) d\Omega = \int_{\Omega} u^*(x, y) \left\{ -\frac{1}{\lambda(u)} \nabla \lambda(u) \nabla u(x) \right\} d\Omega$$  \hspace{1cm} (5)

Integrating by parts, we can derive the following boundary integral equation:

$$c(y) u(y) + \int_{\Gamma} q^*(x, y) u(x) d\Gamma - \int_{\Gamma} u^*(x, y) q(x) d\Gamma$$

$$= -\int_{\Omega} u^*(x, y) \left\{ -\frac{1}{\lambda(u)} \nabla \lambda(u) \nabla u(x) \right\} d\Omega$$  \hspace{1cm} (6)

where $q^*(x, y)$ denotes the normal derivative of fundamental solution:

$$q^*(x, y) = \frac{\partial u^*(x, y)}{\partial n} = \frac{-1}{2\pi} \frac{\partial r}{\partial n}$$  \hspace{1cm} (7)

We shall discuss in the following how to evaluate the domain integral on the right-hand side of Eq. (6).

2.3 Dual reciprocity method (DRM)

In the dual reciprocity formulation, the right-hand side of Eq. (3) is approximated in the following manner:

$$-\frac{1}{\lambda(u)} \nabla \lambda(u) \nabla u(x) = \sum_{j=1}^{N+L} \alpha_j f(x, z^j)$$  \hspace{1cm} (8)

where $N$ is the total number of nodes on the boundary, $L$ the same number in the internal domain, and $\alpha_j$ the unknown coefficient. We have introduced the function
where \( z^j \) is a collocation point of the DRM. It is assumed that \( f(x, z^j) \) is the function of distance \( r^j \) between field point \( x \) and DRM collocation point \( z^j \). In the DRM or MRM[10], various kinds of function are proposed, for example, linear radial basis functions, compactly supported radial basis functions or thin-plate spline.

Substituting Eq. (8) into Eq. (3), we can approximate the governing differential equation as

\[
\nabla^2 u(x) = \sum_{j=1}^{N+L} \alpha_j \nabla^2 \hat{u}(x, z^j) \tag{9}
\]

where \( \hat{u}(x, z^j) \) is the particular solution which satisfies

\[
\nabla^2 \hat{u}(x, z^j) = f(x, z^j) \tag{10}
\]

From the above approximation, we can obtain the following boundary integral equation:

\[
c(y)u(y) + \int_{\Gamma} q^*(x, y)u(x) d\Gamma - \int_{\Gamma} u^*(x, y)q(x) d\Gamma
= \sum_{j=1}^{N+L} \alpha_j \left\{ c(y)\hat{u}(x, z^j) + \int_{\Gamma} q^*(x, y)\hat{u}(x, z^j) d\Gamma - \int_{\Gamma} u^*(x, y)\hat{q}(x, z^j) d\Gamma \right\} \tag{11}
\]

where \( \hat{q}(x, z^j) \) is the normal derivative of the particular solution \( \hat{u}(x, z^j) \), i.e.

\[
\hat{q}(x, z^j) = \frac{\partial \hat{u}(x, z^j)}{\partial n} \tag{12}
\]

From Eq. (11), it is seen that the domain integral is transformed into a series of boundary integrals. If we place the source point on every nodal point on the boundary and in the internal domain, we obtain \( N + L \) equations, which can be summarized into the following matrix form:

\[
Hu - Gq = [H\hat{U} - G\hat{Q}]\alpha \tag{13}
\]

Now, we shall consider how to determine the vector \( \alpha \) in Eq. (13). The function \( f(x, z^j) \) can be evaluated if the two points \( x \) and \( z^j \) are given. We have to determine the unknown coefficients \( \alpha_j \).

Eq. (8) can be rewritten as

\[
\sum_{j=1}^{N+L} \alpha_j f(x, z^j) = -\frac{1}{\lambda(u)} \left\{ \frac{\partial\lambda(u)}{\partial x_1} \frac{\partial u(x)}{\partial x_1} + \frac{\partial\lambda(u)}{\partial x_2} \frac{\partial u(x)}{\partial x_2} \right\} \tag{14}
\]
From the above equation, the unknown vector $\alpha$ is expressed in the following matrix form:

$$\alpha = F^{-1} \left[ K_1 u_{x_1} + K_2 u_{x_2} \right]$$

(15)

where $F^{-1}$ is the inverse matrix of $F$ calculated from the radial basis function introduced, $K_m$ is the coefficient matrix relating to the thermal conductivity, and $u_{x_m}$ is the vector corresponding to the derivative of temperature. There are three ways to evaluate $\partial u(x)/\partial x_m$. That is, using the finite difference approximation, differentiating the interpolated function of $u(x)$, and using the approximation function $f(x, z^j)$. In this study, we shall evaluate $\partial u(x)/\partial x_m$ using the approximation function $f(x, z^j)$. This procedure is explained in the following.

The temperature $u(x)$ is interpolated by using the approximation function $f(x, z^j)$ as follows:

$$u(x) = \sum_{j=1}^{N+L} \beta_j f(x, z^j)$$

(16)

where $\beta_j$ is an unknown coefficient and $f(x, z^j)$ is the approximation function. Differentiating the above equation, we have

$$\frac{\partial u(x)}{\partial x_m} = \sum_{j=1}^{N+L} \beta_j \frac{\partial f(x, z^j)}{\partial x_m}$$

(17)

Applying Eqs. (16) and (17) to all the collocation points, we can evaluate the $u_{x_m}$ as follows:

$$u_{x_m} = F_{x_m} F^{-1} u$$

(18)

Using the above approximation, we can express the vector $\alpha$ as

$$\alpha = F^{-1} \left[ K_1 F_{x_1} + K_2 F_{x_2} \right] F^{-1} u$$

(19)

and the derivative of thermal conductivity $\lambda(u)$ as

$$\frac{\partial \lambda(u)}{\partial x_m} = \frac{\partial \lambda(u)}{\partial u} \frac{\partial u(x)}{\partial x_m}$$

(20)

The derivative of temperature can be evaluated via the approximation function, and then $K_m$ is expressed as follows:

$$K_m = JF_{x_m} F^{-1} u$$

(21)
where $J$ denotes the coefficient of thermal conductivity. Substituting Eq. (19) into Eq. (13), we obtain the following equation:

$$Hu - Gq = Ru \tag{22}$$

where

$$R = [H\hat{U} - G\hat{Q}]F^{-1}[K_1F_{x_1} + K_2F_{x_2}]F^{-1} \tag{23}$$

It is interesting to note that $R$ in Eq. (22) includes the unknown temperature $u(x)$ and hence the system of equations are nonlinear.

The number of equations in Eq. (22) is $N + L$. From this equation, $N$ unknown values on the boundary and $L$ unknown temperatures in the domain can be obtained under the prescribed boundary conditions.

### 3 Numerical examples and consideration

To illustrate usefulness of the proposed DRBEM, a couple of numerical examples are analyzed by using the computer code developed in this study. It is interesting to note that in our formulation any type of temperature dependency can be treated. We shall consider a square domain ABCD in the thick-walled circular cylinder shown in Figure 1. In this model, the thermal conductivity is assumed to change depending on temperature. The boundary is discretized into quadratic elements, and 32 points are assumed as shown in Figure 1. The boundary conditions along AB and CD are such that temperature is prescribed, and the edges BC and AD are subject to a prescribed heat flux. In numerical treatment, these boundary conditions are given by the exact solution of the problem in the thick-walled circular cylinder abcd.

The thermal conductivity is assumed to change as follows:

$$\lambda(u) = A + Bu + Cu^2 \tag{24}$$

In Example 1, we assume that $A = 56.083$, $B = 0.022$, and $C = 0$, whereas in Example 2, $A = 75.341$, $B = -0.078$, and $C = 2.554 \times 10^{-5}$. The temperature-dependency of thermal conductivity is shown in Figure 2.

We shall apply the following approximation function[6]:
Figure 1: Analysis model

Figure 2: Temperature-dependency of thermal conductivity
Figure 3: Results on temperature

Figure 4: Results on heat flux
where \( a \) denotes the influence radius of approximating function. In the following examples, we apply \( a = 1 \) so that all the collocation points in the domain of the problem are assumed to influence each other.

The exact solution is obtained by solving the governing differential equation. To obtain the exact solution of the thick-walled circular cylinder, we apply the Kirchhoff transformation[5]:

\[
v = \int_0^u \lambda(u) du
\]

Introducing the above transformation, Eq. (1) is expressed by

\[
\nabla^2 v = 0
\]

Solving the above linear equation, we can obtain the solution as follows:

\[
C_1 \ln r + C_2 = \int_0^u \lambda(u) du
\]

where the unknown coefficients \( C_1 \) and \( C_2 \) in the above expression can be determined by applying the boundary conditions. The exact solution in this paper denotes the thus-obtained solution.

The results of temperature along the boundary ABCD are shown in Figure 3. It can be seen that accurate results are obtained by the present DRBEM. The results of heat flux which shown in Figure 4 are also fairly accurate. Although we have used \( a = 1 \) in the present computation, we can also use another value of the radius \( a \) smaller than 1. It can be seen that there is probably an optimal value of \( a \) for a particular problem. It can be recommended as future work to find such an optimal value.

4 Conclusion

In this study, a dual reciprocity BEM has been proposed for the steady-state heat conduction problem of temperature-dependent materials. The details of the formulation were presented, and a computer code was developed. Through numerical computation of some examples and comparison with the exact solution, it is revealed that the proposed DRBEM can be successfully applied to the steady-state heat conduction problem.

As future work, we may recommend further investigations of the numerical aspects of the proposed DRBEM, and in addition application to more practical problems, in particular, three-dimensional problems.
References


