A non-singular hybrid boundary element formulation incorporating a higher order fundamental solution

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Abstract

On the basis of the hybrid displacement boundary element method (HDBEM) a non-singular formulation is presented. Although singular fundamental solutions are used for the approximation of domain variables, just regular boundary integrals have to be evaluated. This is achieved by locating the load points outside of the domain. The influence of the load point placement is examined and a preferred region is found leading to good results and a low sensitivity on the actual load point location. The non-singular formulation is applied to 3D elastostatics and elastodynamics. Symmetric and time-independent system matrices are obtained which can be used for the calculation of forced vibrations as well as for eigen-analysis and transient solutions. The domain integral describing inertia forces is mapped on the boundary by means of an analytical higher order fundamental solution. Numerical results are demonstrated for a slender beam in bending.

1 Introduction

In order to efficiently couple the symmetric FEM with the boundary element method (BEM) a symmetric BEM formulation is needed – the hybrid displacement boundary element method (HDBEM) fulfills this requirement. Contrary to the conventional boundary element method, the HDBEM is not based on a boundary integral equation (BIE), it rather uses generalized variational principles as a starting point [5, 6]. The independent field variables are the displacements in the domain as well as the displacements and the tractions on the boundary. The domain field is approximated by a
weighted superposition of fundamental solutions, where the weighting factors are unknowns. The displacement and traction fields on the boundary are interpolated by shape functions multiplied by nodal data. The common property of these trial functions is separation of space and time dependency. The time dependent variables are the nodal values of boundary tractions and displacements as well as the weighting factors of the static fundamental solutions. The space dependency is therefore condensed in the shape functions and the fundamental solutions. Using these approximations, a symmetric system of equation for unknown boundary displacements is obtained by integration over the discretized boundary. Singular integration is avoided completely by not collocating load points and discretization nodes. This is achieved by moving the load points outside the domain, away from the discretization nodes. A non-singular formulations (with respect to boundary integration) is obtained, although singular fundamental solutions are used [7].

Based on the non-singular concept, the equation of motion is derived in section 2 for linear elastodynamics. Due to time-invariant system matrices this equation can be used for forced and transient vibration analyses as well as for eigenvalue problems. The calculation of the mass matrix is discussed in detail in section 3. The domain integral over inertia effects is analytically transformed to the boundary, similar to the transformation used in [3, 4]. Numerical examples validate the proposed non-singular HDBEM formulation for a transient vibrating slender beam in section 4.

2 Non-singular HDBEM formulation for elastodynamics

For a linear-elastic continuum Hamilton’s principle requires stationarity of the functional [10]

$$
\Pi_H(u_i) = \int_{t_1}^{t_2} \left[ \int_{\Omega} \left( -\frac{1}{2} \rho \ddot{u}_i \dot{u}_i + \frac{1}{2} \sigma_{ij} \varepsilon_{ij} - b_i u_i \right) \, d\Omega - \int_{\Gamma_t} \tilde{t}_i u_i \, d\Gamma \right] \, dt
$$

$$
= \text{stat.}
$$

(1)

with the displacement boundary condition $u_i = \ddot{u}_i$ on $\Gamma_u$ and initial conditions. Traction boundary conditions $\tilde{t}_i$ are prescribed on $\Gamma_t$. For the boundary $\Gamma = \Gamma_u \cup \Gamma_t$ holds. The functional contains the kinetic energy of the domain $\Omega$ with mass density $\rho$, the strain energy and the work of body forces $b_i$ and tractions $\tilde{t}_i$ on $\Gamma_t$. The strain tensor $\varepsilon_{ij}$ and the stress tensor $\sigma_{ij}$ are both functions of the displacement vector $u_i$. Therefore, Hamilton’s principle is a single-field functional of the independent variable $u_i$. This principle is generalized by decoupling the domain displacements $u_i$ and the displacements $\ddot{u}_i$ on the boundary $\Gamma$. Adding $\int_{\Gamma} \lambda_i (\ddot{u}_i - u_i) \, d\Gamma = 0$ to (1), compatibility of the independent variables $u_i$ and $\ddot{u}_i$ is enforced in a weak sense by Lagrange multipliers $\lambda_i$. Stationarity of this multi-field
functional leads to Euler equations for a vanishing first variation with independent variables $u_i, \ddot{u}_i, \lambda_i$. Physically the Lagrange multiplier $\lambda_i$ can be interpreted as the traction vector $\vec{t}_i$, leading to a 3-field principle [6]

$$
\Pi_{\text{HD}}(u_i, \ddot{u}_i, \vec{t}_i) = \int_{t_1}^{t_2} \left[ \int_{\Omega} \left( \frac{1}{2} \rho \ddot{u}_i + \frac{1}{2} \sigma_{ij} \varepsilon_{ij} - b_i u_i \right) \, d\Omega - \int_{\Gamma_t} \dot{t}_i u_i \, d\Gamma + \int_{\Gamma} \left( \ddot{u}_i - u_i \right) \, d\Gamma \right] \, dt \equiv \text{stat.} \quad (2)
$$

Stationarity requires that the first variation vanishes, $\delta \Pi_{\text{HD}} = 0$. The first variation is reformulated neglecting body forces $b_i$, integrating by parts the inertia term with respect to time ($\delta \dot{u}_i(t_1) = \delta \dot{u}_i(t_2) = 0$) and applying the divergence theorem to the strain energy term [6]

$$
\delta \Pi_{\text{HD}} = \int_{t_1}^{t_2} \left[ + \int_{\Omega} \left\{ \rho \ddot{u}_i - \sigma_{ij,j} \right\} \delta u_i \, d\Omega + \int_{\Gamma} \left\{ \sigma_{ij} n_j - \ddot{t}_i \right\} \delta u_i \, d\Gamma \\
+ \int_{\Gamma} \left\{ \ddot{u}_i - u_i \right\} \delta \dot{t}_i \, d\Gamma + \int_{\Gamma_t} \left\{ \ddot{u}_i - u_i \right\} \delta \dot{t}_i \, d\Gamma \right] \, dt = 0 \quad (3)
$$

The basic equation for the non-singular HDBEM formulation is obtained.

For the field approximation in the domain, fundamental solutions are used. But contrary to regular BEM and HDBEM formulations the load points of fundamental solutions are not collocated with the nodes of the boundary discretization. Instead, the load points are moved outside the domain. This approach leads to advantages: no domain modification has to be used in order to completely include or exclude the singular points. Thus, the geometry-dependent evaluation of so-called integral free terms is avoided. Even though singular fundamental solutions are applied, the integration over the discretized boundary deals only with regular integral kernels. This is why the presented method is called non-singular. Similar non-singular approaches are presented in [2, 3]. Based on the boundary discretization the load point coordinates are generated. The load point placement according to

$$
\xi_i = x_i + \eta \, D \cdot d'_i \quad (4)
$$

gives very good results [9]. The load point coordinate vector $\xi_i$ depends on the coordinate vector of the node $x_i$, a global scaling factor $\eta$, the diameter $D$ of the largest element adjacent to the particular node and the normalized vector $d'_i$ giving the direction into which the load point is moved with respect...
to the particular node
\[ d_i' = \frac{d_i}{|d_i|}, \quad \text{with} \quad d_i = \sum_{e=1}^{E} n_i^e. \] (5)

The normal vector \( n_i^e \) of the element \( e \) is evaluated at the node of interest. The number of local elements adjacent to this selected node is \( E \). The scaling factor \( \eta \) should be chosen from the interval \( 1.5 \leq \eta \leq 2.5 \) in order to get best numerical results [9]. This interval is obtained by numerical experiments and corresponds well with the results in [2, 3].

The separation of space- and time-dependence [8] is now applied to elastodynamics. The domain field is approximated by superposition of static fundamental solutions

\[ u_i(x_i, t) = \mathbf{u}^T(x_i) \gamma(t), \quad (6) \]
\[ t_i(x_i, t) = t^T(x_i) \gamma(t), \quad (7) \]

where \( x_i \) is a domain point, \( \mathbf{u}^T \) and \( t^T \) are fundamental solution matrices containing the well-known Kelvin fundamental solutions, weighted by time-dependent fictitious loads \( \gamma(t) \). Differentiation of (6) with respect to time gives

\[ \ddot{u}_i(x_i, t) = \mathbf{u}^T(x_i) \dot{\gamma}(t). \] (8)

The fundamental solutions do not change in a variational sense. Therefore, the first variation of (6) is given by

\[ \delta u_i(x_i, t) = \mathbf{u}^T(x_i) \delta \gamma(t). \] (9)

On the boundary space-dependent polynomial shape-functions \( \Phi \) are multiplied by time-dependent nodal data. Thus, the boundary field approximation is formally equal to the domain approximation

\[ \tilde{u}_i(x_i, t) = \Phi^T(x_i) \tilde{u}(t), \quad (10) \]
\[ \tilde{t}_i(x_i, t) = \Phi^T(x_i) \tilde{t}(t). \quad (11) \]

The first variations are

\[ \delta \tilde{u}_i(x_i, t) = \Phi^T(x_i) \delta \tilde{u}(t), \quad (12) \]
\[ \delta \tilde{t}_i(x_i, t) = \Phi^T(x_i) \delta \tilde{t}(t). \quad (13) \]

Traction boundary conditions \( \tilde{t}_i \) are approximated similar to (11). Substituting (6)–(13) in (3), the space-independent variables \( \tilde{u}(t), \tilde{t}(t) \) and \( \gamma(t) \) can be extracted from the spatial integrals. Using static fundamental solutions and positioning the load points outside the domain, the domain
integral over the stress divergence $\sigma_{ij,j}$ vanishes. The matrix definition of the static HDBEM formulation [5] is extended to elastodynamics

\[
N = \int_\Omega \rho \mathbf{u}^* \mathbf{u}^{*\mathbf{T}} d\Omega, \quad F = \int_\Gamma \mathbf{t}^* \mathbf{u}^{*\mathbf{T}} d\Gamma, \\
G = \int_\Gamma \mathbf{u}^* \Phi^{\mathbf{T}} d\Gamma, \quad L = \int_\Gamma \Phi \Phi^{\mathbf{T}} d\Gamma.
\]  \tag{14}

Using (14) in the basic functional (3) leads to three Euler equations

\[
\ddot{\gamma}^\mathbf{T} N + \gamma^\mathbf{T} F - \ddot{\mathbf{t}}^\mathbf{T} G^\mathbf{T} = 0, \\
\gamma^\mathbf{T} G - \ddot{\mathbf{u}}^\mathbf{T} L = 0, \\
\ddot{\mathbf{t}}^\mathbf{T} L - \ddot{\mathbf{t}}^\mathbf{T} L = 0.
\]  \tag{15-17}

Contrary to singular BEM formulations, the calculation of the system matrices according to (14) requires no limiting process since all integral kernels are regular. The matrices $N$, $F$ and $L$ are symmetric. Using (17) the vector of equivalent nodal forces is obtained

\[
\mathbf{q} = L^\mathbf{T} \ddot{\mathbf{t}} = L^\mathbf{T} \ddot{\mathbf{t}}.
\]  \tag{18}

(16) is solved for the fictitious loads. Differentiation with respect to time gives also

\[
\gamma = \left(G^\mathbf{T}\right)^{-1} \ddot{\mathbf{u}} = \mathbf{R} \ddot{\mathbf{u}}, \quad \Rightarrow \quad \ddot{\gamma} = \mathbf{R} \ddot{\mathbf{u}},
\]  \tag{19}

with the definition $\mathbf{R} = \left(G^\mathbf{T}\right)^{-1} \mathbf{L}$. Solving (15) for boundary tractions $\ddot{\mathbf{t}}$ and substituting (18) and (19), leads to a symmetric equation of motion

\[
M \ddot{\mathbf{u}} + K \ddot{\mathbf{u}} = \mathbf{q}, \quad \text{with} \quad M = R^\mathbf{T} NR, \quad K = R^\mathbf{T} FR.
\]  \tag{20}

The mass and stiffness matrix are time-independent. Therefore, this equation system can be used for forced and transient vibration analyses as well as for solving eigenvalue problems. Due to symmetry properties, FEM solvers can be adopted. After solving (20) for boundary displacements $\ddot{\mathbf{u}}(t)$, the fictitious loads are calculated with (19). (6) and (7) lead to displacements and tractions within the domain. Besides symmetry this efficient field point evaluation without additional boundary integration is the most important advantage of HDBEM compared to conventional BEM.

3 Mass matrix

Inertia terms are obviously domain loads and therefore, the mass matrix $M$ requires a domain integration, which also can be seen in definition (14) of the
kernel matrix $N$. To keep a boundary only formulation, a transformation to the boundary is needed. For this task a so-called higher order fundamental solution is used. The derivation of this solution is described first in [9].

For each load point combination $(p, q)$ a submatrix $n_{pq}$ is defined by

$$n_{pq} = n_{ij}^{pq} = \int_{\Omega} \rho u_{li}^* u_{lj}^* d\Omega .$$  \hspace{1cm} (21)

For the sake of clarity the load point indices $p$ and $q$ are omitted in the following discussion. The domain integral is analytically transformed to the boundary by using an auxiliary stress field $\sigma_{ijkl}^a$ (higher order fundamental solution) with the property $\sigma_{ijkl}^a \equiv \rho u_{li}^*$. For the derivation of this higher order fundamental solution $\sigma_{ijkl}^a$ the analogy to the derivation of a real-valued frequency domain fundamental solution for elastodynamics is applied. The frequency domain solution is defined by [9]

$$\sigma_{ij,k} + \rho \omega^2 u_i = 0$$  \hspace{1cm} (22)

Using the Galerkin vector technique a Taylor series for this fundamental solution can be developed with respect to frequency $\omega$ [9]

$$u_{ij}^a(\omega) = \frac{A}{2 c_1^2 r} \left[(c_2^2 - c_1^2) r, r, j - (c_1^2 + c_2^2) \delta_{ij}\right]$$

$$+ \frac{A r}{8 c_1^4 c_2^2} \left[(c_2^4 - c_1^4) r, r, j + (c_1^4 + 3 c_1^4) \delta_{ij}\right] \omega^2$$

$$+ \frac{A r^3}{144 c_1^6 c_2^4} \left[3 (c_1^6 - c_2^6) r, r, j - (5 c_1^6 + c_1^6) \delta_{ij}\right] \omega^4 + \ldots ,$$ \hspace{1cm} (23)

where $c_1$ and $c_2$ are the wave speeds of the longitudinal wave and the equi-voluminal wave, respectively. Choosing the constant $A = (4 \pi \mu)^{-1}$, the first term corresponds to the static fundamental solution. Setting $\omega = 1$, the second term gives the displacement $u_{ij}^a$ of the higher order fundamental solution, satisfying the definition equation $\sigma_{ijkl} = \rho u_{ij}^*$

$$u_{ij}^a = \frac{-r \rho}{128 \pi \mu^2 (1 - \nu)^2} \left[(3 - 4 \nu) r, r, j - (13 - 28 \nu + 16 \nu^2) \delta_{ij}\right].$$ \hspace{1cm} (24)

Differentiation and the material law (linear elastic behavior) leads to

$$\sigma_{ijk} = \frac{\rho}{64 \pi \mu (1 - \nu)^2} \left[(3 - 4 \nu) r, r, j r, k + (5 - 12 \nu + 8 \nu^2) \right.$$

$$\left(r, j \delta_{ik} + r, i \delta_{jk}\right) - (3 - 8 \nu + 8 \nu^2) r, k \delta_{ij}\right],$$ \hspace{1cm} (25)

$$t_{ij}^a = \sigma_{ijk} n_k = \frac{\rho}{64 \pi \mu (1 - \nu)^2} \left[(3 - 4 \nu) \frac{\partial r}{\partial n} r, r, j + (5 - 12 \nu + 8 \nu^2) \right.$$

$$\left(\frac{\partial r}{\partial n} \delta_{ij} + r, i n_j\right) - (3 - 8 \nu + 8 \nu^2) r, j n_i\right].$$ \hspace{1cm} (26)
It is important to mention that the displacements as well as the stresses of the higher order fundamental solution are regular functions and therefore cause no problems in the numerical integration to be performed later.

Using \( \sigma_{ik_i}^{a_i} \equiv \rho u_{t_i}^{*} \), applying the differentiation rule \((a b)_i = a_i b + a b_i\) and Gauss’ theorem to the obtained first domain integral, leads to

\[
\sum_{ij} n_{ij} = \int_{\Omega} \sigma_{ik_i}^{a_i} u_{t_j}^{*} \, d\Omega = \int_{\Omega} \left( \sigma_{ik_i}^{a_i} u_{t_j}^{*} \right)_k \, d\Omega - \int_{\Omega} \sigma_{ik_i}^{a_i} u_{t_j}^{*} \, d\Omega
\]

\[
= \int_{\Gamma} u_{t_j}^{*} \sigma_{ik_i}^{a_i} n_k \, d\Gamma - \int_{\Omega} u_{t_j,k}^{*} C_{lkmn} u_{m_l,n}^{a_i} \, d\Omega. \quad (27)
\]

The stress tensor is replaced by \( \sigma_{ik_i}^{a_i} = C_{lkmn} u_{m_l,n}^{a_i}, \) where \( C_{lkmn} \) is the symmetric elasticity tensor of a linear elastic material. Since the material properties of the auxiliary field are equal to those of the fundamental solution field, one obtains \( \sigma_{mn,j}^{*} = C_{lkmn} u_{ij,k}^{*} \). Therefore the remaining domain integral has a similar structure to the first integral in (27). Again, using the differentiation rule and Gauss’ theorem leads to

\[
\sum_{ij} n_{ij} = \int_{\Gamma} u_{t_j}^{*} \sigma_{ik_i}^{a_i} n_k \, d\Gamma - \int_{\Gamma} u_{m_i}^{a_i} \sigma_{mn,j}^{*} n_n \, d\Gamma + \int_{\Omega} \sigma_{mn,j}^{*} n_{m_l,n} \, d\Omega. \quad (28)
\]

The still remaining domain integral vanishes since the load points of the fundamental solution field are positioned outside the domain \( \Omega \). Using Cauchy’s theorem, (28) is reformulated leading to two regular boundary integrals

\[
\sum_{ij} n_{ij}^{pq} = \int_{\Gamma} u_{t_j}^{*} (\tilde{t}_{ij}^{p}) q \, d\Gamma - \int_{\Gamma} (u_{t_i}^{a_i}) q \, \tilde{t}_{ij}^{p} \, d\Gamma, \quad (29)
\]

where the load point indices are included again. Thus, the presented transformation makes (20) a boundary only formulation.

4 Numerical examples

![Boundary element model of slender beam with 184 linear quadrilateral elements.](image)

To demonstrate the accuracy of the proposed method, a numerical example of a slender beam, pinned at both ends, is presented and compared
with the corresponding analytical Bernoulli beam solution. The vibration of the beam is initiated by an imposed initial deflection out of rest. The static deflection is composed of superimposed mode shapes. The model of the beam is depicted in Figure 1.

The transient solution is calculated using the standard Newmark time-integration scheme [1] with a time increment of $\Delta t = 1\mu s$. For 4 time instants $t$, the transient solution is shown in Figure 2. The plots display the propagation of the bending deflection $u$ from left to right. The dotted line is the analytical beam solution, whereas the solid line represents the numerical HDBEM result.

![Figure 2: Transient solution plotted along the axis of the beam.](image)

### 5 Conclusions

The paper presents a new symmetric non-singular boundary element formulation for elastodynamics based on a mixed variational principle. In earlier published HDBEM formulations, discretization nodes and load points are collocated. Since this collocation is not mandatory, the load points are moved outside the domain. Thus, singular integration is completely avoided. The mass matrix is calculated by an analytical transformation of
the domain integral to the boundary. A time-invariant symmetric equation of motion is obtained. The applicability of the proposed method to elastodynamic problems is validated by a numerical example.

References


