Integral equation method for cylindrical shell under axisymmetric loads

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Abstract

This paper is concerned with the development of the integral equation method for the analysis of a cylindrical shell under axisymmetric loads. The governing equations of the shell are traditionally described as a set of two ordinary differential equations with two unknown variables. These equations are normalized by eliminating their first derivatives, and then multiplied by a weighting function that is a selected Green’s function. Finally, they are repeatedly integrated by parts until their differential operator is shifted from acting on the state variables to the weighting function. Consequently, the differential equations are transformed into a set of integral equations. To complete the analysis procedures, efforts are made to insert various boundary conditions of a shell into the kernels of the integral equations, and to express the internal forces, moments, and displacements of a shell in terms of the state variables. Thus, the integrals are readily available for the analysis of a cylindrical shell. A structure is selected to demonstrate how to apply the method in shell analysis. Three different types of construction are used for the shell; one has a constant thickness, the second has piecewise constant thickness and the third has piecewise linearly varying thickness. The solutions so obtained are compared with the corresponding ones found by the finite element method. They are also compared against the theoretical solutions whenever possible. A good agreement is attained.

1 Introduction

In the mathematical formulation of the physical engineering problems the governing equations of the problem may be expressed either in the form of a differential equation or an integral equation. Integral equation is often mistakenly felt to be either mathematically more advanced or of less practical
usage than differential equation. This situation is changed and an increasing interest in the integral equation has been aroused. Harb and Fu [1,2] presented procedure to transform the governing equations; a system of two ordinary differential equations, into a system of integral equations by using the weak formulation and the integration by parts, and inserting the boundary conditions into the kernels of the integrals and using the integral equations in the analysis and optimum design of a spherical shell. Harb [3] extended this approach for a conical shell.

In this paper, for cylindrical shell, the same procedures presented by Harb [3] are followed and the governing equations are also transformed from ordinary differential equations into integral equations. Various boundary conditions are inserted in the integrals. Thus, the formulation of the integral equations of a cylindrical shell is complete. This set of integral equations is used in the analysis of a cylindrical shell.

2 Governing equations

The internal forces of the shell are shown in Figure 1.

![Figure 1. Internal forces and moments: Axis s (1) is along the straight generator; Axis θ (2) is along the direction of the parallel.](image)

The governing equations for such a cylindrical shell are given as follows, (Mollmann [4], Timoshenko [5]):

\[
\begin{align*}
\psi''' - \frac{h'}{h} \psi' &= - F(s) - \frac{Eh}{R^2} \chi' \\
\chi'' + 3 \frac{h'}{h} \chi' &= \frac{\psi}{D}
\end{align*}
\]  

(1, 2)

in which the notations prime and double prime denote the first and second derivatives with respect to s, and

\[
\chi = w', \quad \psi = \cdot Q_1 \quad \text{and} \quad D = \frac{Eh^3}{12(1-\nu^2)}
\]  

(3, 4, 5)

Where s = the distance from the top of the cylindrical shell measured along the straight generator (meridian); \(\nu\) = the Poisson's ratio; \(E\) = the modulus of elasticity; \(w\) = the normal displacement; \(u\) = the tangential displacement; \(Q_1\) = the shear force on the plane perpendicular to the direction of the meridian; \(h\) = the thickness of the shell which is a function of s and F(s) is the a function of the external loads \(p_1(s)\) and \(p_2(s)\). The internal moments; \(M_1\) and \(M_2\) and the internal forces; \(N_1\) and \(N_2\) are shown in Figure 1. \(W\) and \(U\) are the horizontal and vertical components of the displacements.
3 Method of analysis

Eqns (1) and (2) are two ordinary differential equations with two unknown state variables, $\chi$ and $\psi$. The state variables are functions of the independent variable, $s$. Each of the differential equations may be transformed to an integral equation by using the weak formulation and the integration by parts, Brebbia [6]. The transformation procedure may be described as follows:

1. Normalization of the differential equation. The two equations may be normalized by eliminating their first derivatives. This step is accomplished by changing the state variables, Hill [7]. As a result of this change, eqns (1) and (2) become

\[
\psi^{*\prime\prime} + \alpha(s) \psi^{*} = \frac{F(s)}{\alpha(s)} - \frac{A(s)}{\alpha(s)} \frac{E h}{R^2} \chi^{*}
\]

(6)

\[
\chi^{*\prime\prime} + \beta(s) \chi^{*} = \frac{\alpha(s)}{\beta(s)} \frac{\psi^{*}}{D}
\]

(7)

in which

\[
\psi^{*} = \psi / \alpha(s) \quad \text{and} \quad \chi^{*} = \chi / \beta(s)
\]

(8,9)

\[
\alpha(s) = \exp \left\{-0.5 \int_{s_o}^{s} \left( - \frac{h'(t)}{h(t)} \right) dt \right\} = \left\{ \frac{h(s)}{h(s_o)} \right\}^{0.5}
\]

(10)

\[
\beta(s) = \exp \left\{-0.5 \int_{s_o}^{s} \left( \frac{3 h'(t)}{h(t)} \right) dt \right\} = \left\{ \frac{h^3(s_o)}{h^3(s)} \right\}^{0.5}
\]

(11)

\[
a^{*}(s) = -\frac{3}{4} \left( \frac{h'}{h} \right)^2 + \frac{1}{2} \frac{h''}{h} \quad \text{and} \quad b^{*}(s) = -\frac{3}{4} \left( \frac{h'}{h} \right)^2 - \frac{3}{2} \frac{h''}{h}
\]

(12,13)

2. Translation of the integration interval. For convenience integration, the integration interval $(s_o, s_n)$ is translated to $(-1, 1)$ by means of changing the independent variable; $s$ to $s^*$. Their relationships may be described as follows:

\[
s = 0.5(s_n - s_o)(s^* + 1) + s_o \quad \text{and} \quad s^* = \frac{2(s - s_o)}{s_n - s_o} - 1
\]

(14,15)

As a result of making these changes and the application of chain rule, eqns (6) and (7) become eqns (16) and (17), respectively.

\[
\psi^{*\prime\prime}(s^*) + A(s^*) \psi^{*}(s^*) = -F^*(s^*) - A_1(s^*) \chi^*(s^*)
\]

(16)

\[
\chi^{*\prime\prime}(s^*) + B(s^*) \chi^*(s^*) = B_1(s^*) \psi^*(s^*)
\]

(17)

where

\[
A(s^*) = 0.25 (s_n - s_o)^2 a^*(s^*) \quad \text{and} \quad B(s^*) = 0.25 (s_n - s_o)^2 b^*(s^*)
\]

(18,19)

\[
A_1(s^*) = 0.25 (s_n - s_o)^2 \frac{\beta(s^*) E h(s^*)}{\alpha(s^*) R^2}
\]

(20)
\[ B_1 (s^*) = 0.25 \left( s_n - s_0 \right)^2 \frac{\alpha(s^*)}{\beta(s^*)D(s^*)} \]  

\[ F^*(s^*) = 0.25 \left( s_n - s_0 \right)^2 \frac{F(s^*)}{\alpha(s^*)} \]  

Note that notation dot and double dot represent the first and second derivatives with respect to \( s^* \). Due to this transformation, \( F(s) \) becomes \( F(s^*) \) and \( h' \) and \( p_3' \) are replaced by \( h \) and \( P_3 \). When the resulting expression of \( F(s^*) \) is substituted into eqn (22) one obtains:

\[ F^*(s^*) = \frac{(s_n - s_0)}{2\alpha(s^*)} \left\{ \frac{\varphi(s_n - s_0)}{2R} p_1 \right\} - \frac{P_3}{2\alpha(s^*)} \left( \frac{\varphi(s_n - s_0)}{R} + p_3 \right) \]  

3. Application of the integration by parts. The weak formulation and the integration by parts first applied in eqn (16). Both sides of the equation are multiplied by a weighting function; \( \tau \) and then integrated with respect to a dummy variable; \( t^* \), which varies from \(-1\) to \(+1\). This procedure yields

\[ \int_{-1}^{1} [\varphi(t^*) + A(t^*)\psi(t^*)] \tau dt^* = -\int_{-1}^{1} [F(t^*) \tau dt^* - \int \varphi A(t^*) \psi(t^*) dt^*] \]  

Integrating the left-hand side of eqn (24) by parts twice until the differential operator is shifted from acting on \( \psi \) to \( \tau \), one obtains

\[ \int_{-1}^{1} [\varphi(t^*) + A(t^*) \psi(t^*)] \tau dt^* - \int_{-1}^{1} [\varphi A(t^*) \psi(t^*)] \tau dt^* = -\int_{-1}^{1} [F(t^*) \tau dt^* - \int \varphi A(t^*) \psi(t^*) dt^*] \]  

The weighting function; \( \tau \), is chosen to satisfy the following second-order differential equation

\[ -\delta(s^* - t^*) \]  

and the boundary conditions

\[ \tau(-1) = \tau(1) = 0 \]  

where \( \delta \) is a Dirac delta function, and, \( \tau \) is a Green’s function, Butkovskiy [8], is given as

\[ \tau(s^*, t^*) = -0.5 [s^* - t^* + s^* - 1], \quad s^*, t^* \in (-1, 1) \]  

Substituting eqns (26) and (27) into eqn (25) and taken into consideration that

\[ \int_{-1}^{1} \delta(s^*, t^*) \psi(t^*) dt^* = -\psi(s^*) \]  

one obtains

\[ \psi(s^*) = \int_{-1}^{1} \tau(s^*, t^*)[A(t^*) \psi(t^*) + A(t^*) \psi(t^*) + F(t^*) \psi(t^*)] dt^* + \psi(1) \frac{1 + s^*}{2} + \psi(-1) \frac{1 - s^*}{2} \]  

Differentiating eqn (30) with respect to \( s^* \) and using
\[ \tau \phi^*_{s^*}, t^* = -0.5(1 + t^*), \quad \text{if } s^* > t^* \]  
\[ \tau \phi^*_{s^*}, t^* = -0.5(-1 + t^*), \quad \text{if } s^* < t^* \]  
yields
\[ \psi^*_{\theta}(0) = -0.5 \int_{-1}^{1} \left[ A \phi^* \psi^* + A_1 \phi^* \chi^* + F^* \right] dt^* + 0.5 \psi^*(0) - 0.5 \psi^*(-1) \]  
\[ \psi^*_{\theta}(-1) = -0.5 \int_{-1}^{1} \left[ A \phi^* \psi^* + A_1 \phi^* \chi^* + F^* \right] dt^* + 0.5 \psi^*(0) - 0.5 \psi^*(-1) \]  
Similarly, applying the above procedure to eqn (17), one obtains
\[ \chi^*(s^*) = \int_{-1}^{1} \left[ t^* \right] \left[ B \phi^* \chi^* + B_1 \phi^* \psi^* + F^* \right] dt^* + \chi^*(0) \frac{1+s^*}{2} + \chi^*(-1) \frac{1-s^*}{2} \]  
\[ \chi^*(0) = -0.5 \int_{-1}^{1} \left[ B \phi^* \chi^* + B_1 \phi^* \psi^* + F^* \right] dt^* + 0.5 \chi^*(0) - 0.5 \chi^*(-1) \]  
\[ \chi^*(-1) = -0.5 \int_{-1}^{1} \left[ B \phi^* \chi^* + B_1 \phi^* \psi^* + F^* \right] dt^* + 0.5 \chi^*(0) - 0.5 \chi^*(-1) \]  

4 Internal forces, moments, and displacements

The internal forces, moments, and displacements also need to be expressed in terms of \( \psi^*, \chi^*, \) and \( s^* \). This objective may be achieved by substituting \( \psi^*, \chi^* \) with \( \psi^* \) and \( \chi^* \) as defined in eqn (8) to (9), and by using the chain rule to transform the derivatives \( \psi^* \) and \( \chi^* \) into \( \psi_{\theta}^* \) and \( \chi_{\theta}^* \). The result is
\[ N_1(s^*) = -P(s^*) \]  
\[ N_2(s^*) = Re(s^*) \frac{1}{s_n-s_0} \left[ \frac{R(s^*)}{h(s^*)} \psi^*(s^*) + 2Z(s^*) \right] - Rp(s^*) \]  
\[ M_1(s^*) = \frac{D(s^*)}{s_n-s_0} \left[ \frac{3 h(s^*)}{h(s^*)} \chi^*(s^*) - 2 \chi_{\theta}^*(s^*) \right] \]  
\[ M_2(s^*) = -\nu M_1(s^*) \]  
\[ H(s^*) = -\alpha(s^*) \psi^*(s^*) \]  
\[ U(s^*) = U(-1) + 0.5 \int_{-1}^{s} \left[ N_1(s^*) - \nu N_2(s^*) \right] ds^* \]  
\[ W(s^*) = -\frac{R}{Eh(s^*)} \left[ N_2(s^*) - \nu N_1(s^*) \right] \]  

5 Treatment of boundary conditions

It is a great advantage of the integral equation that the boundary conditions may be included in the equations. All end conditions such as fixed end, simply
supported end, free end, and end with restrained movement, can be inserted into the kernels of the integrals. As a result, these boundary conditions become constituent part in the solution of the equations. To illustrate this point, four different boundary conditions are selected to show how to include them in the integral equations.

5.1 Boundary conditions for $\chi^*(-1)$ and $\chi^*(1)$

Eqn (35) is the integral equation containing $\chi^*(-1)$ and $\chi^*(1)$. In this equation, $\chi^*(-1)$ and $\chi^*(1)$ are first evaluated according to the given boundary conditions, and then substituted back into the integral equation. For clarity this work will be presented in two steps; step 1, evaluation; and step 2, substitution.

**Step 1:** $\chi^*(-1)$ and $\chi^*(1)$ are evaluated for the following boundary conditions, 1A and 1B:

1A. Both ends are clamped and therefore no rotation is possible. Thus,

$$\chi^*(-1) = \chi^*(1) = 0$$

(45)

1B. Neither end is clamped. Both ends at $s^* = -1$ and 1 are free to rotate, moments $M_1(-1)$ and $M_1(1)$ are needed to be prescribed at the ends. Note that the end moment may be zero, a given applied moment, or $k\chi^*$ depending, receptively on whether end is free without external moment, free with external moment, or restrained with a coiled spring whose rotational stiffness is $k_r$. The end moments are given as:

$$M_1(-1) = \frac{D(-1) \beta(-1)}{s_n - s_0} \left[ 3 \frac{h(-1)}{h(-1)} \chi^*(-1) - 2 \chi^*(1) \right]$$

(46)

$$M_1(1) = \frac{D(1) \beta(1)}{s_n - s_0} \left[ 3 \frac{h(1)}{h(1)} \chi^*(1) - 2 \chi^*(1) \right]$$

(47)

Where $\beta(-1) = 1$. Eqns (57) and (57a) may be written in simpler form

$$\chi^*(-1) = \lambda_1 \chi^*(1)$$

and

$$\chi^*(1) = \lambda_2 \chi^*(-1)$$

where

$$\lambda_1 = -1.5 \frac{h(-1)}{h(-1)} \quad \text{and} \quad \lambda_2 = -1.5 \frac{h(1)}{h(1)}$$

(50,51)

$$g_1 = -\frac{(s_n - s_0)M_1(-1)}{2D(-1)}$$

and

$$g_2 = -\frac{(s_n - s_0)M_1(1)}{2D(1)\beta(1)}$$

(52,53)

Observing the four equations, eqns (48), (49), (36), and (37), one finds that there are four unknowns: $\chi^*(-1)$, $\chi^*(1)$, $\chi^*(-1)$, and $\chi^*(1)$. Solving the equations for the unknown’s yields:

$$\chi^*(1) = \frac{1}{A} \left[ -0.5(g_1 - I_1) - (-0.5 + \lambda_1)(g_2 - I_2) \right]$$

(54)

$$\chi^*(-1) = \frac{1}{A} \left[ 0.5(g_2 - I_2) - (0.5 + \lambda_2)(g_1 - I_1) \right]$$

(55)

where

$$A = -\lambda_1 \lambda_2 - 0.5(\lambda_1 - \lambda_2)$$

(56)
\[ I_2 = -\frac{1}{2} \int_{-1}^{1} [B(t^*) \chi^*(t^*) - B_1(t^*) \psi^*(t^*)] dt^* \]  
\[ I_1 = -\frac{1}{2} \int_{-1}^{1} [B(t^*) \chi^*(t^*) - B_1(t^*) \psi^*(t^*)] dt^* \]  

**Step 2:** The values of \( \chi^*(-1) \) and \( \chi^*(1) \) for the boundary conditions obtained in step 1 are substituted into eqn (35) which may then written in the following form:

\[ \chi^*(s^*) + \int_{-1}^{1} G_{11}(s^*, t^*) \chi^*(t^*) dt^* - \int_{-1}^{1} G_{12}(s^*, t^*) \psi^*(t^*) dt^* = f_2(s^*) \]  

\[ G_{11}(s^*, t^*) = \left( 0.5 s^* - t^* + d_1 s^* t^* + d_2 s^* + d_3 t^* + d_4 \right) B(t^*) \]  

\[ G_{12}(s^*, t^*) = \left( 0.5 s^* - t^* + d_1 s^* t^* + d_2 s^* + d_3 t^* + d_4 \right) B_1(t^*) \]  

\[ f_2(s^*) = d_5 + d_6 s^* \]  

in which the constants \( d_1 \) and \( d_6 \) are listed in Table 1.

<table>
<thead>
<tr>
<th>B. Cond.</th>
<th>End Cond.</th>
<th>( d_1 )</th>
<th>( d_2 )</th>
<th>( d_3 )</th>
<th>( d_4 )</th>
<th>( d_5 )</th>
<th>( d_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( s^* = -1 )</td>
<td>( s^* = 1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1A</td>
<td>Clam.</td>
<td>Clam.</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>-0.5</td>
<td>0</td>
</tr>
<tr>
<td>1B</td>
<td>Not Clamp.</td>
<td>Not Clamped</td>
<td>0.5 + ( \frac{\lambda_1 - \lambda_2}{4\Delta} )</td>
<td>( \lambda_1 + \lambda_2 )</td>
<td>( \frac{1}{4\Delta} \left[ 2 - \frac{1}{\lambda_1 + \lambda_2} \right] )</td>
<td>( \frac{1}{2\Delta} \left[ g_2 (1 - \lambda_1) - g_2 (1 + \lambda_1) \right] )</td>
<td>( \frac{1}{2\Delta} (\lambda_2 g_2) )</td>
</tr>
</tbody>
</table>

5.2 Boundary condition for \( \psi^*(-1) \) and \( \psi^*(1) \)

Similarly, \( \psi^*(-1) \) and \( \psi^*(1) \) may be evaluated for the following boundary conditions 2A; end \( s^* = -1 \) is free to move horizontally but end \( s^* = 1 \) is not and 2B; neither end is free to move. Then the values of \( \psi^*(-1) \) and \( \psi^*(1) \) for each boundary condition are inserted into eqns (30), which is written as follows:

\[ \psi^*(s^*) + \int_{-1}^{1} G_{21}(s^*, t^*) \psi^*(t^*) dt^* + \int_{-1}^{1} G_{22}(s^*, t^*) \chi^*(t^*) dt^* = f_2(s^*) \]  

where

\[ G_{21}(s^*, t^*) = k_1(s^*, t^*) A(t^*) \]  

and \( G_{22}(s^*, t^*) = k_1(s^*, t^*) A_1(t^*) \)  

\[ f_2(s^*) = -\int_{-1}^{1} k_1(s^*, t^*) F^*(t^*) dt^* + Z_1(s^*) \]
\[ k(s^*, t^*) = \left(0.5 \left| s^* - t^* \right| + e_1 s^* t^* + e_2 s^* + e_3 t^* + e_4 \right) \]  
(67)

and \( Z_i(s^*) \) and the constants \( e_1 \) to \( e_4 \) are dependent on the boundary conditions on both ends.

### 6 Process of numerical solution

Eqns (59) and (63) are the governing integral equations. This system of integral equations is solved by using Nyström method, Baker [9], which consists essentially of replacing the unknown under the integral sign by a polynomial, integrating this polynomial over an interval and then evaluating the integral at certain specified points within the interval of the integration. This method is a straightforward, generally applicable quadrature technique. In this method the integrals are approximated by the closed Gauss-Chebyshev quadrature rule, Baker [9]. If \( N_q \) quadrature points, are established along the shell meridian, and the governing equations are evaluated at these points, namely \( s_i, i = 1, \ldots, N_q \), then eqn (59) may be expressed as:

\[
\chi^*(s_i) + \sum_{j=1}^{N_q} w_j G_{11}(s_i, t_j) \chi^*(t_j) = \sum_{j=1}^{N_q} w_j G_{12}(s_i, t_j) \psi^*(t_j) = f_1(s_i) \]  
(68)

where \( w_j \) are the weights of the quadrature rule. Eqn (68) can be written in the following matrix form

\[
\begin{bmatrix} I + L^{(1)} \end{bmatrix}_{ij} \chi^*(s_i) - \begin{bmatrix} M^{(1)} \end{bmatrix}_{ij} \psi^*(t_j) = f^{(1)}_i 
\]  
(69)

in a similar transformation to matrix form eqn (63) becomes

\[
\begin{bmatrix} L^{(2)} \end{bmatrix}_{ij} \chi^*(s_i) - \begin{bmatrix} I + M^{(2)} \end{bmatrix}_{ij} \psi^*(t_j) = f^{(2)}_i 
\]  
(70)

Combining eqns (68) and (69), one obtains

\[
\begin{bmatrix} I + L^{(1)} & -M^{(1)} \\ L^{(2)} & I + M^{(2)} \end{bmatrix} \begin{bmatrix} \chi^* \\ \psi^* \end{bmatrix} = \begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix} 
\]  
(71)

where \( I \) is the identity matrix, and

\[
\begin{align*}
\begin{bmatrix} I + L^{(1)} \end{bmatrix}_{ij} &= w_j G_{12}(s_i^*, t_j^*) \\
\begin{bmatrix} M^{(1)} \end{bmatrix}_{ij} &= w_j G_{21}(s_i^*, t_j^*) \\
\begin{bmatrix} L^{(1)} \end{bmatrix}_{ij} &= w_j G_{11}(s_i^*, t_j^*) \\
\begin{bmatrix} L^{(2)} \end{bmatrix}_{ij} &= w_j G_{22}(s_i^*, t_j^*) \\
\end{align*} 
\]  
(72-75)

\[
\begin{align*}
\begin{bmatrix} f^{(1)} \end{bmatrix}_i &= f_1(s_i^*) \\
\begin{bmatrix} f^{(2)} \end{bmatrix}_i &= f_2(s_i^*) = -\sum_{j=1}^{N_q} w_j k_1(s_j^*, t_j^*) E^*(t_j^*) + Z_i(s_i^*) \\
\end{align*} 
\]  
(76-77)

in which \( i = 1 \) and \( j = 1, \ldots, N_q \). The locations and the weights of the quadrature points for the Gauss-Chebyshev rule are

\[
w_i = \frac{2}{n} \left[ 1 - 2 \sum_{k=1}^{n/2} \frac{1}{4k^2 - 1} \cos \frac{2\pi(i-1)k}{n} \right] 
\]  
(78)

\[
s_i^* = t_i^* = \cos \pi(i-1)/n 
\]  
(79)
where $n = N_q - 1$. Eqns (71) are $2N_q$ linearly independent equations that may be solved by Gauss elimination method for the $2N_q$ unknowns $\chi_j$ and $\psi_j$. These unknowns are the state variables at the quadrature points $s_j$. Consequently, the internal forces, stresses, and displacements can be found.

7 Theoretical solution

When the thickness of the shell is constant, the controlling differential equations of the cylindrical shell may be solved by theoretical means. The governing differential equations are first reduced to a fourth order differential for one unknown, $w$, the normal displacement. The complete general solution of the fourth order differential equation has four arbitrary constants defined from the boundary conditions at both ends. Since the method has been described in details elsewhere, it will not be repeated here.

8 Engineering application

In this section the proposed integral equation method is applied for the solution of a practical engineering structure solved by Timoshenko [5] and shown by Figure 2. It is a concrete cylindrical water tank. The concrete has a modulus of elasticity, $E$ equals 3120 ksi ($215 \times 10^7$ kN/m$^2$) and a Poisson’s ratio, $v$ equals 0.25. The weight per unit volume is 0.03613 lb/in$^3$ (10 kN/m$^3$). The shell is solved with three different types of construction:

1. The thickness is constant throughout the structure, $h=14.0$ in.
2. The thickness is piecewise constant in each design segment (Table 2).
3. The thickness is piecewise linearly variable in each segment (Table 2).

![Figure 2. Concrete cylindrical water tank.](image)

Table 2. Thickness for Problems 2 and 3.

<table>
<thead>
<tr>
<th>s (in)</th>
<th>h, Problem 2 (in)</th>
<th>h, Problem 3 (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>112</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>172</td>
<td>14</td>
<td>13</td>
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<tr>
<td>252</td>
<td>16</td>
<td>15</td>
</tr>
<tr>
<td>312</td>
<td>16</td>
<td>17</td>
</tr>
</tbody>
</table>

In this computation, 41 quadrature points are arbitrarily selected for Problem 1 and 47 quadrature points and four design segments are also arbitrarily chosen for Problems 2 and 3. They can be changed easily if there is need to do so. The flexibility in the selection of the number of the quadrature points and the number of design segments is a good feature of the integral method.

The integral equation method and the various cases of the boundary conditions are coded in a Fortran program.
9 Results, comparison, comprehension and conclusions

For comparison purposes, the finite element method is applied on the three problems. Theoretical method is also used when possible. A finite element program called AXSH3 written by Johnston, Weaver and Johnston [10], was adopted for this purpose. Twenty elements are used for Problem 1 and 23 elements for Problems 2 and 3. Since in this program, each element has two quadrature points, the total number of the quadrature points is comparable to that used in the integral equation method. For Problem 1, the thickness of the shell is constant; it is solved theoretically by Timoshenko [5]. These results and the results of the integral equation method are presented in the same graph for each type of computation to facilitate an easy comparison. The results are presented in Figures 3-6. Only sample results are included herein due to space limitation. A study of the Figures 3 and 4, which contain the results of shells with constant thickness, reveals that there is an excellent agreement between the theoretical, finite element and integral equation solutions. Figure 5 contains the meridian flexural stresses of shell of Problem 2 and Figure 6 presents the vertical displacement for Problem 3. Only the results of the finite element and the integral equation methods are presented and compared. They exhibit a good agreement. In conclusion, the advantages of this method may be summarized as follows:

1. The numerical required by the integral equation method are relatively simple and straightforward.

2. The boundary conditions such as fixed end, simply supported end, free end, and ends with constrained movements are inserted into the kernels of the integrals. Thus, they become an integral part in the shell analysis.

3. The input data required by this method are merely simple.

Thus, the integral equation method is an alternative good technique for shell analysis.

![Figure 3. Parallel membrane stresses for Problem 1.](image1.png)

![Figure 4. Horizontal displacement, W, for Problem 1.](image2.png)
Figure 5. Meridian flexural stresses for Problem 2.

Figure 6. Vertical Displacement, U, for Problem 3.

References


