Boundary element stress analysis of plates on 2 parameter foundation under generalized loading

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Abstract

This paper introduces boundary integral equations for bending moments and shear forces for thick Reissner plates on two parameter elastic foundation. The reduction of domain loading integrals to boundary integrals or simple terms, is demonstrated in those integral equations, for cases with generalized loading, which include concentrated loading, and line loading acting on arbitrary curves defined on the plate surface. Several case studies with different boundary and loading conditions, have been analysed and results were compared with corresponding analytical solutions. It has been shown that the new formulations lead to very accurate solutions for bending moments and shear forces at different points inside the plate.

1 Introduction

The boundary element analysis of thick Reissner plates on elastic foundation has been approximated in the literature ([1], [2]). The author and his co-workers [3] have introduced boundary element derivations for the actual thick Reissner plates on two parameter elastic foundation. The fundamental solution of this type of problems is usually expressed in terms of modified Bessel functions $K_\nu(z)$ and $K_{\nu}(z)$, with $z$ being a real or complex variable, and the kernel functions containing $\log z$, $1/z$, and $1/z^2$ singularities. In a recent publication [4], the author has introduced a modified fundamental solution, which leads to the elimination of most singular terms, and with simplified kernel functions for concentrated and distributed loading. The non-singular fundamental solution has been generalized [5] to include different possible values for characteristic equation roots. Simplification of domain loading integrals has also been demonstrated. The stresses were calculated using finite difference techniques.
This paper introduces boundary integral equations for bending moments and shear forces based on previously published fundamental solutions. The reduction of domain loading integrals to boundary integrals or simple terms, is demonstrated in those integral equations, for cases with generalized loading, which include concentrated loading, and line loading acting on arbitrary curves defined on the plate surface. Several case studies with different boundary and loading conditions, have been analysed and results were compared with corresponding analytical solutions. It has been shown that the new formulations lead to very accurate solutions for bending moments and shear forces at different points inside the plate.

2 Review of governing equations

For a plate of uniform thickness $h$, resting on a two parameter elastic foundation with parameters $k$, $G_p$, the equilibrium equations over the plate thickness derived according to Reissner’s theory [6], can be written in the following form:

$$
\begin{align*}
M_{11,1} + M_{21,2} - Q_1 &= 0 , \quad M_{12,1} + M_{22,2} - Q_2 = 0 , \\
Q_{1,1} + Q_{2,2} + q - k w + G_p \nabla^2 w &= 0
\end{align*}
$$

where bending moments and shear forces, per unit length, are defined in terms of lateral deflection and average slope angles as follows:

$$
\begin{align*}
M_{a \beta} &= \frac{1}{2} (1 - \nu) D (\theta_{a, \beta} + \theta_{\beta, a}) \\
&\quad + \delta_{a, \beta} [D \nu (\theta_{1,1} + \theta_{2,2}) + \zeta (q - k w + G_p \nabla^2 w)] \\
Q_{\beta} &= \frac{1}{2} (1 - \nu) D \lambda^2 (\theta_{\beta} + w_{,\beta})
\end{align*}
$$

where $D$ is the flexural rigidity of the plate, $\nu$ is Poisson’s ratio, $\lambda^2 = 10h^2/3$, 

$$
\zeta = \frac{\nu}{(1 - \nu) \lambda^2}, \quad f_{,a} = \frac{\partial f}{\partial x_a}, \quad a = 1, 2, \quad \beta = 1, 2,
$$

and 

$$(x_1, x_2) \equiv (x, y), \quad f_1 \equiv f_x, \quad f_2 \equiv f_y,$$

Equation (2) can also be written as follows:

$$
\begin{align*}
M_{a \beta} &= \frac{1}{2} (1 - \nu) D (\theta_{a, \beta} + \theta_{\beta, a}) \\
&\quad + D \nu \delta_{a, \beta} [\beta_1 (\theta_{1,1} + \theta_{2,2}) + \beta_2 (q - k w)]
\end{align*}
$$

where 

$$
\begin{align*}
\beta_1 &= 1 - G_p / [(1 - \nu) D \lambda^2 + 2 G_p], \quad \beta_2 = 1 / [(1 - \nu) D \lambda^2 + 2 G_p]
\end{align*}
$$

Using a weighted-residual approach [7], the boundary integral equations with
at an internal point \((x_i, y_i)\) the slope angles and lateral deflection are given by:

\[
\theta_x(x_i, y_i) = -\oint_{\Gamma} \left( T_{11} \theta_n + T_{21} \theta_t + T_{31} w \right) \, d\Gamma \\
+ \oint_{\Gamma} \left( U_{11} M_n + U_{21} M_{nt} + U_{31} Q_n \right) \, d\Gamma + \iint_{\Omega} p_1 \, q \, dx \, dy
\]

\[
\theta_y(x_i, y_i) = -\oint_{\Gamma} \left( T_{12} \theta_n + T_{22} \theta_t + T_{32} w \right) \, d\Gamma \\
+ \oint_{\Gamma} \left( U_{12} M_n + U_{22} M_{nt} + U_{32} Q_n \right) \, d\Gamma + \iint_{\Omega} p_2 \, q \, dx \, dy
\]

\[
w(x_i, y_i) = -\oint_{\Gamma} \left( T_{13} \theta_n + T_{23} \theta_t + T_{33} w \right) \, d\Gamma \\
+ \oint_{\Gamma} \left( U_{13} M_n + U_{23} M_{nt} + U_{33} Q_n \right) \, d\Gamma + \iint_{\Omega} p_3 \, q \, dx \, dy
\]

where the kernel functions; \(U_{ij}, T_{ij}, p_j\) are listed in Appendix A.

3 Boundary equations for bending moments and shear forces

Substituting from Equations (6)-(8) into (4), then the boundary integral equations of bending moments at an internal point \((x_i, y_i)\) can be expressed as follows:

\[
M_{\alpha\beta}(x_i, y_i) = \delta_{\alpha\beta} D \nu \beta_2 \, q + \oint_{\Gamma} \left( A_{\alpha\beta 1} \theta_n + A_{\alpha\beta 2} \theta_t + A_{\alpha\beta 3} w \right) \, d\Gamma \\
- \oint_{\Gamma} \left( B_{\alpha\beta 1} M_n + B_{\alpha\beta 2} M_{nt} + B_{\alpha\beta 3} Q_n \right) \, d\Gamma - \iint_{\Omega} L_{\alpha\beta} \, q \, dx \, dy
\]

where

\[
A_{\alpha\beta} = \frac{1}{2} D \left( 1 - \nu \right) \left( T_{j\alpha, \beta} + T_{j\beta, \alpha} \right) + \delta_{\alpha\beta} D \nu \left[ \beta_1 \left( T_{j1, 1} + T_{j2, 2} \right) + \beta_2 k T_{j3} \right]
\]

\[
B_{\alpha\beta} = \frac{1}{2} D \left( 1 - \nu \right) \left( U_{j\alpha, \beta} + U_{j\beta, \alpha} \right) + \delta_{\alpha\beta} D \nu \left[ \beta_1 \left( U_{j1, 1} + U_{j2, 2} \right) + \beta_2 k U_{j3} \right]
\]

\[
L_{\alpha\beta} = \frac{1}{2} D \left( 1 - \nu \right) \left( p_{\alpha, \beta} + p_{\beta, \alpha} \right) + \delta_{\alpha\beta} D \nu \left[ \beta_1 \left( p_{1, 1} + p_{2, 2} \right) + \beta_2 k p_3 \right]
\]

and \(j = 1, 2, 3\).

Similarly by substituting from Equations (6)-(8) into (3), the boundary integral equations of shear forces at an internal point \((x_i, y_i)\) can be expressed as follows:
\[ Q_\beta(x_i, y_i) = \int \int \left( \varphi_{\beta_1} \partial_n + \varphi_{\beta_2} \partial_i + \varphi_{\beta_3} \psi \right) d\Gamma - \int \int \left( \psi_{\beta_1} M_n + \psi_{\beta_2} M_{nt} + \psi_{\beta_3} Q_n \right) d\Gamma - \int \int \Lambda_\beta q d\Gamma (15) \]

where
\[ \varphi_{\beta_j} = \frac{1}{2} D \lambda^2 (1 - \nu) \left( T_{j,3,\beta} - T_{j,\beta} \right) \]

\[ \psi_{\beta_j} = \frac{1}{2} D \lambda^2 (1 - \nu) \left( U_{j,3,\beta} - U_{j,\beta} \right) \]

\[ \Lambda_\beta = \frac{1}{2} D \lambda^2 (1 - \nu) \left( p_{3,\beta} - p_{\beta} \right) \]

Using the expressions of kernel functions of displacement boundary integral equations given in Appendix A, explicit expressions for the moment and shear kernel functions can be obtained as listed in Appendix B.

### 4 Analysis of loading domain integrals

Using the approach presented in Reference 8, the loading domain integrals in different boundary integral equations presented in this work can be simplified and reduced for different types of loading as will be discussed in this section.

#### 4.1 Case with concentrated loading

Consider a case where a concentrated shear force \( F \) and bending moments \( T_x \) and \( T_y \) are acting at a point \((x_l, y_l, z_l)\), in the \( z, x, \) and \( y \) directions, respectively. Using the properties of the Dirac delta function, it can be proved that:

\[ \int \int p_j q d\Gamma \] \( \Omega \)

\[ = \int \int p_j \left( -T_x \partial_d + T_y \partial_d + F \right) \delta(x \cdot x_i, y \cdot y_i) d\Gamma \]

\[ = \left( T_x \partial_d - T_y \partial_d + F \right) \] \( p_j \) \( \) at \( x = x_l, y = y_l \) \( 17 \)

\[ = -Q_{j1} T_y + Q_{j2} T_x + Q_{j3} F \]

where
\[ Q_{j\beta} = \frac{\partial}{\partial x_\beta} p_j, \quad Q_{j3} = p_j \] \( \) at \( x = x_l, y = y_l \)

Hence, it can be proved that:

\[ \int \int L_{\alpha\beta} q d\Gamma \] \( \Omega \)

\[ = -C_{\alpha\beta 1} T_x + C_{\alpha\beta 2} T_x + C_{\alpha\beta 3} F \] \( 18 \)

where
\[ C_{\alpha\beta j} = \frac{1}{2} D (1 - \nu) \left( Q_{\alpha j, \beta} + Q_{\beta j, a} \right) \]

\[ + \delta_{\alpha\beta 3} D \nu \left( \beta_1 (Q_{1,1} + Q_{2,2}) + \beta_2 Q_{3,1} \right) \] \( 19 \)
Similarly, it can be deduced that:

$$\int \int \Lambda_B q \, dx \, dy = -C_{\beta 1}^s T_y + C_{\beta 2}^s T_x + C_{\beta 3}^s F$$

where

$$C_{\beta j}^s = \frac{1}{2} D \lambda^2 (1 - v) (Q_{3 j, \beta} - Q_{\beta j})$$

(21)

4.2 Case of line loading on an arbitrary curve

Consider a loaded curve $\Gamma_i$ inside the domain $\Omega$ of the plate midplane, with line loadings $Q, M_n, M_r$, which represent shear force, and normal and tangential bending moments per unit length along $\Gamma_i$. Using an approach similar to that given in Reference 8, it can be shown that:

$$\int \int p_j q \, dx \, dy = \int_{\Gamma_i} \left( M_n \frac{\partial}{\partial t} - M_r \frac{\partial}{\partial n} + Q \right) p_j \, d\Gamma$$

$$= \int_{\Gamma_i} \left( -q_{j1} M_t + q_{j2} M_n + q_{j3} Q \right) d\Gamma$$

(22)

where

$$q_{j3} = p_j, \quad q_{j\beta} = \frac{\partial}{\partial n} p_j$$

Hence it can be deduced for this case of loading that:

$$\int \int I_{\alpha \beta} q \, dx \, dy = \int_{\Gamma_i} \left( -\chi_{\alpha \beta 1} M_t + \chi_{\alpha \beta 2} M_n + \chi_{\alpha \beta 3} Q \right)$$

(23)

where

$$\chi_{\alpha \beta j} = \frac{1}{2} D \lambda (1 - v) \left( q_{\alpha j, \beta} + q_{\beta j, \alpha} \right)$$

$$+ \delta_{\alpha \beta} D v \left( p_{11} q_{11, j} + q_{22, j} + k p_{22} q_{33} \right)$$

(24)

Similarly, it can be deduced that:

$$\int \int \Lambda_{\beta} q \, dx \, dy = \int_{\Gamma_i} \left( -\chi_{\beta 1}^s M_t + \chi_{\beta 2}^s M_n + \chi_{\beta 3}^s Q \right)$$

(25)

where

$$\chi_{\beta j}^s = \frac{1}{2} D \lambda^2 (1 - v) \left( q_{3 j, \beta} - q_{\beta j} \right)$$

(26)

5 Case studies

The previous derivations were implemented in a computer program for the analysis of thick Reissner plates on two parameter elastic foundation. Several case studies were analysed, and some of the results are presented.

5.1 Clamped circular plate under concentric line loading

This case represents a clamped solid circular plate, with outer radius $= 10$ m, and thickness $= 1$ m is analysed. The material Young's modulus is $1.092 \times 10^7$ N/m²,
Figure 1: Radial distribution of bending moment $M$, for a clamped circular plate under concentric line loading.

Figure 2: Radial distribution of shear force $Q_r$ for a clamped circular plate under concentric line loading.
Figure 3: Radial distribution of bending moment $M_r$ for a simply-supported circular plate under concentric line moment.

Figure 4: Radial distribution of shear force $Q_r$ for a simply-supported circular plate under concentric line moment.
and its Poisson’s ratio = 0.3. The foundation elastic parameters are $k = 2.0 \times 10^4$ N/m$^3$, $G_p = 1.0 \times 10^4$ N/m. Four cases of concentric line loading were tested, with a total shear force $F = 3.1416 \times 10^4$ N, acting uniformly along concentric circles with radii $R_o = 2, 4, 6, 8$ m. The radial distributions of bending moment $M$, and shear force $Q$, as obtained from boundary element analysis were plotted against corresponding analytical solutions, as shown in Figures 1 and 2, respectively. It is clear from those figures that there is an excellent agreement between boundary element results and corresponding analytical solutions.

5.2 Simply-supported circular plate under concentric line moment

This case represents a simply-supported plate with the same dimensions and properties as the previous case. Four cases of concentric line moment were considered, where a total normal bending moment $M_o = 3.1416 \times 10^5$ Nm, was applied uniformly along concentric circles with radii = 2, 4, 6, 8 m. The radial distributions of bending moment $M$, and shear force $Q$, as obtained from boundary element analysis were plotted against corresponding analytical solutions, as shown in Figures 3 and 4. It is clear that a very good agreement between boundary element results and corresponding analytical solutions has been obtained.

6 Conclusions

New derivations for boundary integral equations of bending moments and shear forces of thick plates on two parameter elastic foundation have been introduced and validated for cases with line loading and moments. The new formulations lead to very accurate solutions for bending moments and shear forces at different points inside the plate. Discontinuous distributions of internal bending moments and shear forces, as expected for cases with line loading and line moments, have been accurately evaluated with the new boundary element derivations, using two source points before and after the discontinuity.

References


Appendix A: Kernel functions for displacement boundary integral equations

The kernel functions in eqns (6)-(8) are defined as follows:

\[ U_{ij} = \sum_{l=1}^{3} a_i U^*_l(\lambda_i, r), \quad T_{ij} = \sum_{l=1}^{3} a_i T^*_l(\lambda_i, r), \quad p_j = \sum_{l=1}^{3} a_i p^*_l(\lambda_i, r) \]

where \( a_l = \frac{1}{2\pi a_0 D (\lambda_i^2 - \lambda_j^2)(\lambda_i^2 - \lambda_j^2)} \), \((i,j,l) = (1,2,3)\) at different orders.

The parameters \( \lambda_1, \lambda_2, \lambda_3 \) are defined as follows:

\[ \lambda_1^2, \lambda_2^2 = B \pm \sqrt{B^2 - C}, \quad \lambda_3 = \lambda \]

where \( a_0 = 1 + \left( \frac{2 - \nu}{1 - \nu} \right) \left( \frac{G_p}{\lambda^2 D} \right), \quad 2B = \frac{1}{a_0} \left[ \left( \frac{2 - \nu}{1 - \nu} \right) \left( \frac{k}{\lambda^2 D} \right) + \frac{G_p}{D} \right] \).

\( C = kl/(a_0 D) \), and \( U^*_{ij}(z), \quad T^*_{ij}(z), \quad p^*_j(z) \) are defined in terms of differential expressions of \( K_0(cr) \), where \( z = cr, \quad c = \lambda_j \), and listed as follows:

\( U^*_{\alpha\beta} = \left( \tilde{r} \cdot \tilde{r} \right) \left[ c_1 K_0(z) + c_2 B_1(z) \right] - c_2 \frac{\partial}{\partial n_\alpha} \frac{\partial}{\partial x_\beta} A_1(z) \)

\( U^*_{\alpha3} = -c_31 \frac{\partial}{\partial n_\alpha} K_1(z), \quad U^*_{3\beta} = c_32 \frac{\partial}{\partial x_\beta} K_1(z), \quad U^*_{33} = c_33 K_0(z) \)
\[ T_{1\beta}^* = -c \left[ \beta \frac{\partial r}{\partial n} + v(\hat{\beta} \cdot \hat{\beta}) \frac{\partial r}{\partial t} \right] F(z) + \frac{1}{D} \frac{\partial r}{\partial n} K_1(z) \]

\[ + c_2 \frac{\partial r}{\partial x_\beta} \left[ gK_1(z) + (4g - v - 1) \frac{A_1(z)}{z} \right] \]

\[ T_{2\beta}^* = c(1 - v) \left\{ c_2 \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\beta} E(z) - \frac{1}{2} \left[ \beta \frac{\partial r}{\partial t} + (\hat{\beta} \cdot \hat{\beta}) \frac{\partial r}{\partial n} \right] F(z) \right\} \]

\[ T_{3\beta}^* = \frac{(1 - v)\lambda^2}{2} \left\{ c_2 \left[ c_1 K_0(z) + c_2 B_1(z) + \frac{\gamma c^2 (c^2 - \lambda^2)}{\lambda^2} K_1(z) \right] \right\} \]

\[ - \left[ c_2 + \frac{\gamma c^2 (c^2 - \lambda^2)}{\lambda^2} \right] \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\beta} A_1(z) \right\} \]

\[ T_{13}^* = a_3 \left[ (1 + v) \frac{K_1(z)}{z} - gA_1(z) \right] + \frac{G_p}{D} a_2 K_0(z) \]

\[ T_{23}^* = (1 - v) a_3 \frac{\partial r}{\partial n} \frac{\partial r}{\partial t} A_1(z), \quad T_{33}^* = -c a_4 \frac{\partial r}{\partial n} K_1(z) \]

\[ p_{1\beta}^*(z) = -a_1 \frac{\partial}{\partial x_\beta} K_0(z), \quad p_{2\beta}^* = a_2 K_0(z) \]

where \( z = cr, a = 1, 2, \beta = 1, 2, (x_1, x_2) = (x, y), (\hat{1}, \hat{2}) = (\hat{x}, \hat{y}) \),

\[ \hat{n}_1 = \hat{n} = l\hat{x} + m\hat{y}, \quad \frac{\partial r}{\partial n_1} = \frac{\partial r}{\partial n} \]

\[ \hat{n}_2 = \hat{x} = -m\hat{x} + l\hat{y}, \quad \frac{\partial r}{\partial n_2} = \frac{\partial r}{\partial t} \]

\[ \alpha_1 = (c^2 - \lambda^2) \left[ \frac{v c^2}{(1 - v) \lambda^2} + 1 \right] \quad \alpha_3 = c^2 (c^2 - \lambda^2) \]

\[ \alpha_2 = (c^2 - \lambda^2) \left[ \frac{2 - v}{(1 - v) \lambda^2} - 1 \right] \quad \alpha_4 = \alpha_3 + \alpha_2 \frac{G_p}{D} \]

\[ \phi = \frac{2}{D \lambda^2 (1 - v)} \quad \gamma = \lambda^2 + \frac{v \phi k}{(1 - v)} \]

\[ c_1 = (c^2 - \lambda^2) \left[ c^2 (1 + \phi G_p) - \phi k \right] \quad \gamma = \gamma + \lambda^2 \phi G_p \]

\[ c_2 = c^2 \left[ \frac{(1 + v) + \phi G_p - \phi k}{(1 - v)} + \lambda^2 \right] \quad c_3 = c_1 + \alpha_1 \phi k, \]
\[ c_{31} = c (c^2 - \lambda^2) \]
\[ c_{32} = \frac{c}{\lambda^2} c_{31} \]
\[ c_{33} = (c^2 - \lambda^2) \left[ \frac{2c^2}{(1 - \nu) \lambda^2} - 1 \right] + \alpha_2 \varphi G_p \]
\[ g = \left( \frac{\partial r}{\partial n} \right)^2 + \nu \left( \frac{\partial r}{\partial t} \right)^2 \]
\[ A_1(z) = K_o(z) + 2K_1(z)/z \]
\[ B_1(z) = K_o(z) + K_1(z)/z \]
\[ F(z) = (c_1 + c_2)K_1(z) + 2c_2 A_1/z \]
\[ E(z) = K_1(z) + 4A_1/z. \]

**Appendix B: Kernel functions for internal moment and shear boundary integral equations**

The kernel functions in eqns (9) and (13) are defined as follows:

\[ A_{ijk} = \sum_{l=1}^{3} a_l A_{ijk}^*(\lambda_i, r), \quad B_{ijk} = \sum_{l=1}^{3} a_l B_{ijk}^*(\lambda_i, r), \quad \Phi_{ij} = \sum_{l=1}^{3} a_l \Phi_{ij}^*(\lambda_i, r), \text{ etc.} \]

and \( A_{ijk}^*(z), B_{ijk}^*(z), \Phi_{ij}^*(z), \Psi_{ij}^*(z) \) are as follows:

\[ A_{\alpha\beta 1}^* = D c^2 (1 - \nu) \left\{ \alpha_1 G_p \left[ \frac{K_1 \delta_{\alpha\beta}}{z} - \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} A_1 \right] - \left[ l_\alpha l_\beta + \nu (\hat{r} \cdot \hat{r}_\alpha) (\hat{r} \cdot \hat{r}_\beta) \right] \frac{E}{z} \right\} \]

\[ + \left( \frac{\partial r}{\partial t} \right)^2 \left[ \frac{l_\alpha}{\partial x_\alpha} + l_\beta \frac{\partial r}{\partial x_\beta} \right] + \nu \left( \frac{\partial r}{\partial t} \right)^2 \left[ (\hat{r} \cdot \hat{r}_\alpha) \frac{\partial r}{\partial x_\alpha} + (\hat{r} \cdot \hat{r}_\beta) \frac{\partial r}{\partial x_\beta} \right] \left( \frac{c_1 + c_2}{2} A_1 + 2 c_2 E \right) \]

\[ - \left( A_1 + A_{\alpha\beta} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right) - (1 + \nu) c_2 \delta_{\alpha\beta} A_1 + c_2 E \left[ \frac{g \delta_{\alpha\beta} - (6g - \nu - 1) \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} }{z} \right] \]

\[ + \delta_{\alpha\beta} D \nu \left\{ \xi_1 K_o \frac{G_p}{D} + c^2 \zeta_1 \left[ g A_1 - (1 + \nu) \frac{K_1}{z} \right] \right\} \]

\[ A_{\alpha\beta 2}^* = \frac{1}{2} D c^2 (1 - \nu)^2 \left\{ \left[ l_\alpha (\hat{r} \cdot \hat{r}_\beta) + l_\beta (\hat{r} \cdot \hat{r}_\alpha) \right] \left( c_1 + c_2 \right) A_1 + \frac{K_1}{z} + 2 c_2 \frac{A_1}{z^2} \right\} \]

\[ + \left( \frac{\partial r}{\partial t} \right)^2 \left[ \frac{l_\alpha}{\partial x_\alpha} + l_\beta \frac{\partial r}{\partial x_\beta} \right] + \nu \left( \frac{\partial r}{\partial t} \right)^2 \left[ (\hat{r} \cdot \hat{r}_\alpha) \frac{\partial r}{\partial x_\alpha} + (\hat{r} \cdot \hat{r}_\beta) \frac{\partial r}{\partial x_\beta} \right] \left( \frac{c_1 + c_2}{2} A_1 + 2 c_2 A_1 E \right) \]

\[ - 2 c_2 \left( \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} \right) A_1 \left( \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} + E \left( \frac{6}{z} \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} - \delta_{\alpha\beta} \right) \right) \]

\[ + \delta_{\alpha\beta} D \nu (1 - \nu) c^2 \zeta_1 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} A_1 \]
\[
A_{\alpha\beta}^* = - \frac{c (1 - \nu)}{2 \varphi} \left\{ \left( l_0 \frac{\partial r}{\partial x_\beta} + l_\beta \frac{\partial r}{\partial x_\alpha} \right) \left[ (c_1 + c_2) K_1 + 2 (c_2 + c_3) \frac{A_1}{z} \right] \\
- 2 (c_2 + c_3) \frac{\partial r}{\partial n} \left[ \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} - \delta_{\alpha\beta} \right] \frac{A_1}{z} \right\} \\
- \delta_{\alpha\beta} \frac{c v k (c^2 - \lambda^2)^2}{\lambda^2} \frac{\partial r}{\partial n} K_1
\]

\[
B_{\alpha\beta}^* = - D c (1 - \nu) \left\{ \left[ \frac{\partial r}{\partial n_\alpha} \left( \hat{n}_\gamma \cdot \hat{t}_\beta \right) + \frac{\partial r}{\partial n_\beta} \left( \hat{n}_\gamma \cdot \hat{t}_\alpha \right) \right] \left[ \frac{(c_1 + c_2)}{2} + c_2 \frac{A_1}{z} \right] \\
- c_2 \frac{\partial r}{\partial n_\gamma} \left[ E(z) \frac{\partial r}{\partial x_\alpha} - \frac{A_1}{z} \right] \frac{\partial r}{\partial n_\beta} \right\} \\
- \delta_{\alpha\beta} D c v \zeta_1 \frac{\partial r}{\partial n_\gamma} K_1
\]

\[
B_{\alpha\beta3} = D (1 - \nu) c c_{32} \left[ \delta_{\alpha\beta} \frac{K_1}{z} - 2 \frac{\partial r}{\partial x_\alpha} \frac{\partial r}{\partial x_\beta} A_1 \right] + \delta_{\alpha\beta} D c v \zeta_2 K_0(z)
\]

\[
\varphi_{\beta1}^* = \frac{c}{\varphi} \left\{ \left[ (c_1 + c_2) K_1 + 2 (c_2 + c_3) \frac{A_1}{z} \right] \left[ l_\beta \frac{\partial r}{\partial n} + v (\hat{t} \cdot \hat{t}_\beta) \frac{\partial r}{\partial t} \right] \\
- (c_2 + c_3) \frac{\partial r}{\partial x_\beta} \left[ g K_1 + (4 g - 1 - \nu) \frac{A_1}{z} \right] \right\} \\
- (c_2 + c_3) \frac{\partial r}{\partial x_\beta} \left[ G_\rho \frac{\partial r}{\partial x_\beta} K_1 \right]
\]

\[
\varphi_{\beta2}^* = \frac{c (1 - \nu)}{2 \varphi} \left\{ \left[ l_\beta \frac{\partial r}{\partial t} + (\hat{t} \cdot \hat{t}_\beta) \frac{\partial r}{\partial n} \right] \left[ (c_1 + c_2) K_1 + 2 (c_2 + c_3) \frac{A_1}{z} \right] \\
- 2 (c_2 + c_3) E(z) \frac{\partial r}{\partial n} \frac{\partial r}{\partial t} \frac{\partial r}{\partial x_\beta} \right\}
\]

\[
\varphi_{\beta3}^* = \frac{1}{\varphi} \left\{ - \frac{1}{2} (1 - \nu) \lambda^2 l_\beta (c_1 + c_2) K_0(z) \\
+ \left[ c^2 a_4 + \frac{1}{2} (1 - \nu) \lambda^2 (c_1 + c_2) \right] \left[ \frac{\partial r}{\partial n} \frac{\partial r}{\partial x_\beta} A(z) - l_\beta \frac{K_1}{z} \right] \right\}
\]

\[
\psi_{\beta\alpha}^* = \frac{1}{\varphi} \left\{ - (c_1 + c_2) (\hat{n}_\alpha \cdot \hat{t}_\beta) K_0(z) + (c_2 + c_3) \left[ \frac{\partial r}{\partial n_\alpha} \frac{\partial r}{\partial x_\beta} A(z) - (\hat{n}_\alpha \cdot \hat{t}_\beta) \frac{K_1}{z} \right] \right\}
\]

\[
\psi_{\beta3}^* = - \frac{(c_{32} + c_{33} c)}{\varphi} \frac{\partial r}{\partial x_\beta} K_1(z)
\]

where \[ \zeta_1 = \beta_1 c_1 + \beta_2 c_2 + c_3, \zeta_2 = \beta_2 k c_3 - \beta_1 c_2 c_3 \]

and other parameters are as in Appendix A.