Hierarchical plate modelling by boundary elements

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Abstract

We obtain a hierarchical formulation for the differential and integral equations of the twelfth-order plate bending model for homogeneous transversely isotropic materials (Poniatovskii [1], Reissner [2]). The hierarchical specification means that the direct reduction from the twelfth-order equation system furnishes corresponding systems of lower-order: a) the systems of sixth-order of Reissner [3] and Mindlin [4], and b) the system of fourth-order of Kirchhoff. By means of the selection of particular values of two coupling coefficients we obtain the sixth-order system of Reissner. The sixth-order system of Mindlin and the fourth-order system of Kirchhoff are included in the hierarchy by using the tensors of Westphal Jr. et al. [5]. The differential equations for the twelfth-order model are obtained by application of the Hellinger-Reissner mixed variational principle. We show that the system of differential equations gives a best approximation to the three-dimensional elasticity equations (Lur'e [6]) than displacement-based models (Schwab & Wright [7]). The hierarchical reduction of the twelfth-order integral equations furnishes the well known sixth-order equations of van der Weeën [8]. With the corresponding fundamental solutions the BEM can consequently be used to solve the hierarchy of problems.

1 Introduction

The present paper deals with a hierarchical derivation of the differential and integral equations of high-order plate bending models for homogeneous transversely isotropic materials. Concerning the order of the polynomial
used for approximation of the displacement field, a problem will be specified as a \((n_r, n_s)\) plate model, where \(n_r\) refers to the polynomial order of the in-plane displacements, whereas \(n_s\) is the polynomial order of the out-of-plane displacement. For displacement-based approaches the notation \(d-(n_r, n_s)\) will be used, while for approaches based on the mixed Hellinger-Reissner functional the notation \(s-(n_r, n_s)\) will be adopted. The hierarchy of models we are concerned is shown schematically in Figure 1.

![Figure 1: The hierarchy of plate models.](image)

Small Greek and Latin indices and capital Latin indices range over the intervals \(\alpha, \beta, \ldots = \{1, 2\}; \ i, j, \ldots = \{1, 2, 3\}; \) and \(I, J, \ldots = \{1, 2, 3, 4, 5, 6\}\), except when explicitly established, according to the rules of the indicial notation. The position of these indices, whether upper or lower, does not change the rule; the upper indices are enclosed in parentheses for clarity. The equations are treated within the framework of the linearized theory of elasticity.

Let \((\mathbf{x}) := (x_1, x_2, x_3) \in \mathbb{R}^3\) with \((\mathbf{\bar{x}}) := (x_1, x_2) \in \mathbb{R}^2\) be a Cartesian coordinate system. Consider an open multi-connected region \(V(\mathbf{x})\), defined by a constant thickness \(h = 2c > 0\) and a middle surface \(\Omega(\mathbf{\bar{x}})\) with a Lipschitz continuous boundary \(\Gamma(\mathbf{\bar{x}}), \ \bar{\Omega} := \Omega \cup \Gamma\). The geometry of the problem is shown in Table 1.

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<td>(V := {\mathbf{x} \mid \mathbf{\bar{x}} \in \Omega, -c &lt; x_3 &lt; c})</td>
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<td>boundary</td>
<td>(\Gamma)</td>
<td>(S := {\mathbf{x} \mid \mathbf{\bar{x}} \in \Gamma, -c \leq x_3 \leq c})</td>
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Table 1: Geometry of the problem.
2 General 2-D Formulation

The 3-D equilibrium equations are

$$\sigma_{ij,j}(x) + b_i(x) = 0,$$  \hspace{1cm} (1)

where $b_i(x)$ are the volume forces, with loadings on the faces $R_\pm$

$$\sigma_{\alpha 3}(x) |_{x_3=\pm c} := 0 \quad \text{and} \quad \sigma_{33}(x) |_{x_3=\pm c} := \pm \frac{1}{2} q_3(x).$$ \hspace{1cm} (2)

A plate bending problem is specified by the following ansatz: Given a pair $\{r, s\}$, $r, s = 1, 2, \ldots \in \mathbb{N}$, the variation of displacements across the plate thickness is approximated by

$$u_\alpha(x) := \sum_{k=1}^{M} a_{2k-1}\phi^{(k)}_{\alpha}(\overline{x}) c P_{2k-1}(\xi),$$ \hspace{1cm} (3a)

$$u_3(x) := \sum_{k=1}^{M} a_{2(k-1)}\phi^{(k)}_{3}(\overline{x}) \frac{d P_{2k-1}(\xi)}{d \xi},$$ \hspace{1cm} (3b)

where $\phi^{(k)}_{i}(\overline{x})$ are the plate displacements, $\xi := x_3/c$, $M := \max\{r, s\}$, $P_n(\xi)$ are Legendre polynomials of order $n$, and $a_n \in \mathbb{R}$, $n = 0, \ldots, 2M - 1$. Additionally we define $n_r := 2r - 1$ and $n_s := 2(s - 1)$. In this work we consider only the case $r := s$.

The plate stresses are given by

$$\sigma_{\alpha \beta}^{(k)}(\overline{x}) := \int_{-c}^{c} a_{2k-1}\sigma_{\alpha \beta}(x) c P_{2k-1}(\xi) \, dx_3, \quad k = 1, \ldots, r,$$ \hspace{1cm} (4a)

$$\sigma_{\alpha 3}^{(k)}(\overline{x}) := \int_{-c}^{c} a_{2(k-1)}\sigma_{\alpha 3}(x) \frac{d P_{2k-1}(\xi)}{d \xi} \, dx_3, \quad k = 1, \ldots, s,$$ \hspace{1cm} (4b)

$$\sigma_{33}^{(k-r)}(\overline{x}) := \int_{-c}^{c} a_{2k-1}\sigma_{33}(x) c P_{2k-1}(\xi) \, dx_3, \quad k - r = 1, \ldots, s - 1.$$ \hspace{1cm} (4c)

For the transverse plate stresses we use the alternative notations

$$\sigma^{(k)}_\alpha(\overline{x}) \equiv \sigma_{\alpha 3}^{(k)}(\overline{x}), \quad \sigma^{(k-r+1)}(\overline{x}) \equiv \sigma_{33}^{(k-r)}(\overline{x})$$ \hspace{1cm} (5)

The 2-D displacement and stress components are grouped in the following arrays...
We added 1 to the index of the left variable in eqn (5b) and defined
\[ \sigma^{(1)}(x) := q_3(x) \] (7)
and included consequently the loading in \( \tau^{(1)}_3(x) \), eqn (6b).

Considering the equations of the 3-D theory of elasticity it results from the displacement ansatz (3) (see Westphal Jr. [9]):

The distribution of volume forces \( b_i(x) \) along the thickness direction are

\[ b_\alpha(x) = \frac{1}{2c^2} \sum_{k=1}^{M} \frac{4k-1}{a_{2k-1}} m^{(k)}_\alpha(x) P_{2k-1}(\xi) \], \hspace{1cm} (9a)

\[ b_3(x) = \frac{1}{2c} \sum_{k=1}^{M} \frac{1}{a_{2k-1}} m^{(k)}_3(x) \left[ P_{2(k-1)}(\xi) - P_{2k}(\xi) \right] \], \hspace{1cm} (9b)

with the 2-D corresponding forces \( m^{(k)}_i(x) \)

\[ m^{(k)}_\alpha(x) = \int_{-c}^{c} a_{2k-1} b_\alpha(x) c P_{2k-1}(\xi) \, dx_3 \], \hspace{1cm} (10a)

\[ m^{(k)}_3(x) = \int_{-c}^{c} a_{2k-1} b_3(x) \frac{dP_{2k-1}(\xi)}{d\xi} \, dx_3 \]. \hspace{1cm} (10b)
For the fundamental solution 3-D loadings \( b_i^*(x) \), acting on a point \( \bar{x} \equiv P \), are of the same form as given by eqn (9), with plate concentrated loadings \( m_i^{*k}(\bar{x}) \equiv m_i^{*k}(P) \) of the same form as eqns (10). For an arbitrary observation point \( \bar{x} \equiv Q \) it follows

\[
m_i^{*k}(Q) := \delta(P,Q)m_i^{*k}(P),
\]

with a Dirac distribution \( \delta(P,Q) \) with singularity on collocation point \( P \).

### 3 s-(3,2) hierarchical differential equations

Setting \( r := 2 \) (\( r := s \), see above) we obtain from the general formulation the equations of the twelfth-order Poniatovskii/Reissner model. For \( a_0 = a_1 := 1 \), \( a_2 = a_3 := -\frac{7}{2} \), and \( a_5 := \frac{799}{2c^2} \) it results

- **displacements** (Lewiński [10]):

  \[
  u_\alpha(x) = \phi_\alpha^{(1)}(\bar{x})x_3 + \phi_\alpha^{(2)}(\bar{x})\frac{21}{4} \left(1 - \frac{5}{3}\xi^2\right)x_3, \quad (12a)
  
  u_3(x) = \phi_3^{(1)}(\bar{x}) + \phi_3^{(2)}(\bar{x})\frac{21}{4} \left(1 - 5\xi^2\right); \quad (12b)
  
- **stresses** (Reissner [2]):

  \[
  \sigma_{\alpha\beta}(x) = \frac{3}{2c^2} \left[ \sigma_{\alpha\beta}^{(1)}(\bar{x}) + \sigma_{\alpha\beta}^{(2)}(\bar{x}) \left(1 - \frac{5}{3}\xi^2\right) \right] \xi, \quad (13a)
  
  \sigma_{\alpha3}(x) = \frac{1}{8c} \left[ 6\sigma_{\alpha}^{(1)}(\bar{x}) (1 - \xi^2) + \sigma_{\alpha}^{(2)}(\bar{x}) (1 - 6\xi^2 + 5\xi^4) \right], \quad (13b)
  
  \sigma_{33}(x) = \frac{1}{8} \left[ 2\sigma^{(1)}(\bar{x}) (3 - \xi^2) + \sigma^{(2)}(\bar{x}) (1 - 2\xi^2 + \xi^4) \right] \xi; \quad (13c)
  
- **plate stresses**:

  \[
  \sigma_{\alpha\beta}^{(1)}(\bar{x}) := \int_{-c}^{c} \sigma_{\alpha\beta}(x)x_3 \, dx_3, \quad (14a)
  
  \sigma_{\alpha\beta}^{(2)}(\bar{x}) := \int_{-c}^{c} \sigma_{\alpha\beta}(x)\frac{21}{4} \left(1 - \frac{5}{3}\xi^2\right)x_3 \, dx_3, \quad (14b)
  
  \sigma_{\alpha}^{(1)}(\bar{x}) := \int_{-c}^{c} \sigma_{\alpha3}(x) \, dx_3, \quad (14c)
  
  \sigma_{\alpha}^{(2)}(\bar{x}) := \int_{-c}^{c} \sigma_{\alpha3}(x)\frac{21}{4} \left(1 - 5\xi^2\right) \, dx_3, \quad (14d)
  
  \sigma^{(2)}(\bar{x}) := \int_{-c}^{c} \sigma_{33}(x)\frac{693}{16c^2} (15 - 70\xi^2 + 63\xi^4) \, dx_3; \quad (14e)
2-D equilibrium equations:

\[ \sigma_{\alpha\beta}(\bar{\mathbf{x}}) - \sigma^s_{\alpha}(\bar{\mathbf{x}}) + m^s_{\alpha}(\bar{\mathbf{x}}) = 0, \quad \sigma^s_{\alpha}(\bar{\mathbf{x}}) + \sigma^s_{\alpha}(\bar{\mathbf{x}}) = 0; \quad (15) \]

2-D equilibrium equations for the fundamental solution:

\[ \sigma^{s*}_{\alpha\beta}(\bar{\mathbf{x}}) - \sigma^{s*}_{\alpha}(\bar{\mathbf{x}}) + m^{s*}_{\alpha}(\bar{\mathbf{x}}) = 0, \quad (16a) \]

\[ \sigma^{s*}_{\alpha\alpha}(\bar{\mathbf{x}}) + \sigma^{s*}_{\alpha}(\bar{\mathbf{x}}) + m^{s*}_{\alpha}(\bar{\mathbf{x}}) = 0, \quad (16b) \]

with \( \sigma^{s*}(\bar{\mathbf{x}}) = 0 \) (observe that \( \sigma^{(1)}(\bar{\mathbf{x}}) \) is the loading!).

The equations corresponding to the \( s,(1,0) \) sixth-order model of Reissner [3] are obtained by discarding the 2-D variables with an upper index (2). The key idea of the hierarchical derivation is to identify explicitly all the contributions of these variables on the equations that follow.

To obtain a system of 2-D differential equations we use the Hellinger-Reissner variational principle. As constitutive equation we consider a transversely isotropic material with elasticity modulus \( E \), Poisson coefficient \( \nu \) and parameters (see Westphal Jr. et al. [5])

\[ k_E := \sqrt{\frac{E_3}{E}}, \quad k_G := \frac{G_3}{G}, \quad k_\nu := \frac{\nu_3}{\nu}, \quad (17) \]

with \( \nu_3 := \sqrt{\nu_1 \nu_2} \). For isotropy \( k_E = k_G = k_\nu := 1 \). After integration along the plate thickness we obtain the principle as a function of 2-D variables (Reissner [2], Westphal Jr. [9])

\[ -\Pi_R(\tau, \varphi) := \iint_\Omega \left\{ \frac{1}{2(1-\nu^2)} D(\psi) \left[ (1+\nu)\sigma^{(\psi)}_{\gamma\gamma}(\psi) \sigma^{(n)}_{\gamma\gamma} - \nu \sigma^{(\psi)}_{\gamma\gamma}(\psi) \right] \right. \]

\[ -E(\psi)\sigma^{(\psi)}_{\gamma\gamma}(\psi) \sigma^{(n)}(\psi) + \frac{1}{2} G(\psi) \sigma^{(\psi)}_{\gamma\gamma}(\psi) \sigma^{(n)}(\psi) + \frac{1}{2} C(\psi) \sigma^{(\psi)}(\psi) \sigma^{(n)}(\psi) \]

\[ + \left[ \sigma^{(\psi)}_{\gamma\gamma} - \sigma^{(\psi)}(\psi) \right] \varphi^{(\psi)}(\psi) + \left[ \sigma^{(\psi)}_{\gamma\gamma} + \sigma^{(\psi)}(\psi) \right] \varphi^{(\psi)}(\psi) \right\} \right. \]

\[ d\bar{x}_1 d\bar{x}_2 \]

\[ - \int_{\Gamma_1} \psi_{\nu} \psi_{\nu} d\Gamma, \quad (18) \]

where \( \tau \) is a 3 \times 6 matrix with the plate stresses as components, see eqn (6d) for \( M := 2 \), and \( \varphi \) is the vector of plate displacements

\[ \varphi(\bar{\mathbf{x}}) := \left( \varphi^{(1)}_1(\bar{\mathbf{x}}), \varphi^{(1)}_2(\bar{\mathbf{x}}), \varphi^{(1)}_3(\bar{\mathbf{x}}), \varphi^{(2)}_1(\bar{\mathbf{x}}), \varphi^{(2)}_2(\bar{\mathbf{x}}), \varphi^{(2)}_3(\bar{\mathbf{x}}) \right)^T, \quad (19) \]
with the "new" plate displacements resulting from the variational principle

\[ \varphi^{(1)}_{\alpha}(\bar{x}) := \frac{3}{2c^2} \int_{-c}^{c} u_{\alpha}(x) \xi \, dx, \quad (20a) \]

\[ \varphi^{(1)}_{3}(\bar{x}) := \frac{3}{4c} \int_{-c}^{c} u_{3}(x) (1 - \xi^2) \, dx, \quad (20b) \]

\[ \varphi^{(2)}_{\alpha}(\bar{x}) := \frac{3}{2c^2} \int_{-c}^{c} u_{\alpha}(x) \left(1 - \frac{5}{3} \xi^2\right) \xi \, dx, \quad (20c) \]

\[ \varphi^{(2)}_{3}(\bar{x}) := \frac{1}{8c} \int_{-c}^{c} u_{3}(x) (1 - 6\xi^2 + 5\xi^4) \, dx. \quad (20d) \]

With 3-D displacements \( u_i \) as given in (12) it follows \( \varphi^{(s)}_{i} = \varphi^{(d)}_{i} \), showing that 2-D displacements of s-based models are a better representation of a corresponding 3-D problem than d-based models. The 2-D displacements \( \varphi^{(s)}_{i} \) are weighted averages across the thickness, see Reissner [11]. The constitutive constants appearing in eqn (18) are

\[ D^{(11)} := \frac{2Ec^3}{3(1 - \nu^2)}, \quad D^{(12)} = D^{(21)} := 0, \quad D^{(22)} := \frac{21}{4} D^{(11)}, \quad (21a) \]

\[ G^{(11)} := \frac{3}{5G_{3c}}, \quad G^{(12)} = G^{(21)} := \frac{1}{21} G^{(11)}, \quad G^{(22)} := \frac{4}{189} G^{(11)}, \quad (21b) \]

\[ E^{(11)} := \frac{3
u_3}{5c\sqrt{E_3E}}, \quad E^{(12)} = E^{(21)} := \frac{1}{21} E^{(11)}, \quad E^{(22)} := \frac{4}{189} E^{(11)}, \quad (21c) \]

\[ C^{(11)} := \frac{17c}{70E_3}, \quad C^{(12)} = C^{(21)} := \frac{8}{153} C^{(11)}, \quad C^{(22)} := \frac{8}{1683} C^{(11)}. \quad (21d) \]

From the variational principle it follows:

- 2-D differential equations

\[ \sigma^{(\xi)}_{\alpha\beta} = D^{(\xi\eta)} \frac{1 - \nu}{2} \left[ \varphi^{(\eta)}_{\alpha,\beta} + \varphi^{(\eta)}_{\beta,\alpha} + \frac{2\nu}{1 - \nu} \varphi^{(\eta)}_{\gamma,\gamma} \delta_{\alpha\beta} \right] + (1 + \nu) D^{(\xi\eta)} E^{(\eta\gamma)} \sigma^{(\gamma)} \delta_{\alpha\beta}, \quad (22a) \]

\[ \sigma^{(1)}_{\alpha} = D^{(11)} \frac{1 - \nu}{2} \lambda^2 \left[ \varphi^{(1)}_{\alpha} + \varphi^{(1)}_{3,\alpha} - \frac{9}{4} \left( \varphi^{(2)}_{\alpha} + \varphi^{(2)}_{3,\alpha} \right) \right], \quad (22b) \]

\[ \sigma^{(2)}_{\alpha} = 9 D^{(22)} \frac{1 - \nu}{2} \lambda^2 \left[ \varphi^{(2)}_{\alpha} + \varphi^{(2)}_{3,\alpha} - \frac{1}{21} \left( \varphi^{(1)}_{\alpha} + \varphi^{(1)}_{3,\alpha} \right) \right], \quad (22c) \]

\[ \sigma^{(2)} = \frac{1}{C^{(22)}} \left[ E^{(21)} \sigma^{(2)}_{\beta\beta} - C^{(21)} \sigma^{(1)} - \varphi^{(2)}_{3} \right]; \quad (22d) \]
- 2-D equilibrium equations (see eqns (15) with \( m_{0}^{(\xi)} := 0 \))

\[
\sigma_{\alpha\beta,\beta}^{(\xi)} - \sigma_{\alpha}^{(\xi)} = 0 , \quad \sigma_{\alpha,\alpha}^{(\xi)} + \sigma_{\xi}^{(\xi)} = 0 ;
\] (23)

- tractions

\[
t_{\alpha}^{(\xi)} := \sigma_{\alpha\beta}^{(\xi)} n_{\beta} , \quad t_{3}^{(\xi)} := \sigma_{\beta}^{(\xi)} n_{\beta} ;
\] (24)

- boundary conditions

\[
\varphi_{i}^{(\xi)} = \varphi_{i}^{(\xi)} \quad \text{on } \Gamma_{u} , \quad t_{i}^{(\xi)} = t_{i}^{(\xi)} \quad \text{on } \Gamma_{t} ,
\] (25a)

with

\[
t_{\alpha}^{(\xi)} = \sigma_{\alpha\beta}^{(\xi)} n_{\beta} , \quad t_{3}^{(\xi)} = \sigma_{\alpha}^{(\xi)} n_{\alpha} .
\] (25b)

The first coupling variable, related to shear quantities, is given by

\[
k_{R} := \begin{cases} 
1 & \text{for the model } s-(1,0) \\
\frac{28}{25} & \text{for the model } s-(3,2) 
\end{cases},
\] (26)

entering into eqn (22) through the variable

\[
\lambda^{2} \equiv \lambda^{2}(k_{R}, G_{3}, c) := k_{R}k_{G} \frac{5}{2c^{2}}.
\] (27)

The Lamé/Navier equations of the model s-(3,2) are

\[
L_{i,j} u_{j} = -F_{i} \sigma_{1}^{(1)}
\] (28)

with

\[
u := \begin{pmatrix} \varphi_{1}^{(1)} , \varphi_{2}^{(1)} , -\varphi_{3}^{(1)} , \varphi_{1}^{(2)} , \varphi_{2}^{(2)} , -\varphi_{3}^{(2)} \end{pmatrix}^{T}
\] (29a)

\[
F := \begin{pmatrix} B^{(1)} \partial_{1} , B^{(1)} \partial_{2} , F^{(1)} , B^{(2)} \partial_{1} , B^{(2)} \partial_{2} , F^{(2)} \end{pmatrix}^{T},
\] (29b)

\[
L := \begin{bmatrix} L_{A} \& L_{AB} \& L_{B} \end{bmatrix}_{3 \times 3}^{sym}
\] (29c)

The 3 \times 3 sub-matrices above are

\[
L_{A} := D \frac{1 - \nu}{2} \begin{bmatrix}
\Delta - \lambda^{2} + \nu_{A} \partial_{11}^{2} & \nu_{A} \partial_{12}^{2} & \lambda^{2} \partial_{1} \\
\nu_{A} \partial_{12}^{2} & \Delta - \lambda^{2} + \nu_{A} \partial_{22}^{2} & \lambda^{2} \partial_{2} \\
-\lambda^{2} \partial_{1} & -\lambda^{2} \partial_{2} & -\lambda^{2} \Delta
\end{bmatrix}_{sym}
\] (30a)

\[
L_{AB} := \frac{9D}{4} \frac{1 - \nu}{2} \begin{bmatrix}
\lambda^{2} + \nu_{AB} \partial_{11}^{2} & \nu_{AB} \partial_{12}^{2} & (9\eta - \lambda^{2}) \partial_{1} \\
\nu_{AB} \partial_{12}^{2} & \lambda^{2} + \nu_{AB} \partial_{22}^{2} & (9\eta - \lambda^{2}) \partial_{2} \\
-\lambda^{2} \partial_{1} & -\lambda^{2} \partial_{2} & \lambda^{2} \Delta
\end{bmatrix}
\] (30b)

\[
L_{B} := \frac{21D}{4} \frac{1 - \nu}{2} \begin{bmatrix}
\Delta - 9\lambda^{2} + \nu_{B} \partial_{11}^{2} & \nu_{B} \partial_{12}^{2} & 9(\lambda^{2} + \eta) \partial_{1} \\
\nu_{B} \partial_{12}^{2} & \Delta - 9\lambda^{2} + \nu_{B} \partial_{22}^{2} & 9(\lambda^{2} + \eta) \partial_{2} \\
-9\lambda^{2} \partial_{1} & -9\lambda^{2} \partial_{2} & 9(\zeta - \lambda^{2} \Delta)
\end{bmatrix}_{sym}
\] (30c)
where $\partial_\alpha \equiv \frac{\partial}{\partial x_\alpha}$, $\partial_{\alpha \beta}^2 \equiv \frac{\partial^2}{\partial x_\alpha \partial x_\beta}$, and $\Delta \equiv \partial_{\alpha}^2$, with the constants

$$\alpha(\nu, \nu_3) := 63(1 - \nu) - 121\nu_3^2, \quad \beta \equiv \beta(\nu, \nu_3) := 1 - \nu - 2\nu_3^2,$$

$$\gamma \equiv \gamma(\nu, \nu_3) := \frac{297}{5} \frac{\nu_3^2}{\alpha(\nu, \nu_3)},$$

$$\eta \equiv \eta(\nu, \nu_3, k_E, c) := \frac{55(1 + \nu)\nu_3 k_E}{\alpha(\nu, \nu_3) k_G^2} \lambda^2,$$

$$\zeta \equiv \zeta(\nu, \nu_3, k_E, c) := \frac{12375}{28} \frac{(1 - \nu^2) k_E}{\alpha(\nu, \nu_3) k_G^2} \lambda^4,$$

$$B^{(1)} := D^{(11)}(1 + \nu)E^{(11)} \left[ 1 - 33 \frac{\beta(\nu, \nu_3)}{\alpha(\nu, \nu_3)} \right],$$

$$B^{(2)} := D^{(22)}(1 + \nu)E^{(21)} \left[ 1 - 308 \frac{\beta(\nu, \nu_3)}{\alpha(\nu, \nu_3)} \right],$$

$$F^{(1)} := 1, \quad F^{(2)} := -693 \frac{\beta(\nu, \nu_3)}{\alpha(\nu, \nu_3)}, \quad D := D^{(11)}$$

and

$$\nu_A := [1 + \gamma] \nu_\nu, \quad \nu_{AB} := \frac{28}{27} \gamma \nu_\nu, \quad \nu_B := \left[ 1 + \frac{28}{27} \gamma \right] \nu_\nu, \quad \nu_\nu := \frac{1 + \nu}{1 - \nu}.$$ (32)

In order to reduce this system to those corresponding to the s-(1,0) model it is necessary to discard all terms containing variables with upper indices including 2, to set $k_R := 1$ according to eqn (26), and finally to discard all terms containing the variable $\alpha$, eqn (31a), here identified as the second coupling variable. Then, it is easy to obtain now the Lamé/Navier equation of the s-(1,0) model (Westphal Jr. et al. [5]):

$$L_{ij} u_j = -F_i \sigma^{(1)}$$

with

$$u := \begin{pmatrix} \varphi^{(1)}_1, \varphi^{(1)}_2, \varphi^{(1)}_3 \end{pmatrix}^T,$$

$$F := \begin{pmatrix} B^{(1)} \partial_1, \ B^{(1)} \partial_2, \ F^{(1)} \end{pmatrix}^T, \quad B^{(1)} := \frac{\nu k_N k_G}{(1 - \nu) \lambda^2 k_E},$$

$$\lambda := \frac{1 - \nu}{2} \begin{bmatrix} \Delta - \lambda^2 + \nu_\nu \partial_{11}^2 & \nu_\nu \partial_{12}^2 & \lambda^2 \partial_{11} & \lambda^2 \partial_{12} \end{bmatrix}.$$

4 s-(3,2) hierarchical integral equations

The hierarchical boundary integral equations are easily obtained by using the weighted residual method (Brebbia et al. [12]). The Somigliana identi-
ties for a collocation point \( p \in \Gamma \) are

\[
c_{ij}(p)u_i(p) + \int_\Gamma T_{ij}(p,q)u_j(q) \, d\Gamma = \int_\Gamma U_{ij}(p,q)t_j(q) \, d\Gamma + \int_\Omega U_i(p,Q)\sigma^{(1)}(Q) \, d\Omega . \tag{35}
\]

For details see Westphal Jr. [9]. The integral equations for the model \( s-(1,0) \) can be easily obtained in the form (Westphal Jr. et al. [5])

\[
c_{ij}(p)u_i(p) + \int_\Gamma T_{ij}(p,q)u_j(q) \, d\Gamma = \int_\Gamma U_{ij}(p,q)t_j(q) \, d\Gamma + \int_\Omega \left[ U_{i3}(p,Q) - \frac{\nu_3 k_G}{(1-\nu)\lambda^2 k_E} U_{i\alpha,\alpha}(p,Q) \right] \sigma^{(1)}(Q) \, d\Omega . \tag{36}
\]

5 2-D plate versus 3-D elasticity

Our objective is to compare the 2-D differential equations with the corresponding equations of the 3-D elasticity theory for an elastic layer, see e.g. Lu're [6], Schwab & Wright [7] and Gregory [13]. This can be easily performed by the Helmholtz decomposition theorem, with (Lewiński [10])

\[
\Psi(\xi) := \epsilon_{\alpha\beta} \varphi^{(\xi)}_{\alpha\beta} , \quad \Phi(\xi) := \varphi^{(\xi)}_{0,0} , \tag{37}
\]

where \( \epsilon_{\alpha\beta} \) is the permutation tensor. By substituting eqns (37) into the Lamé/Navier system we obtain the solutions for \( \Psi(\xi) \), \( \Phi(\xi) \) and \( \varphi^{(\xi)}_{3} \).

5.1 Solution for \( \Psi(\xi) \)

- Model \( s-(1,0) \) (Reissner [14])

\[
\left[ h^2 \Delta - s-(1,0) S_1^2 \right] \Psi^{(1)} = 0 , \tag{38a}
\]

\[
s-(1,0) S_1 := \sqrt{10 k_G} ; \tag{38b}
\]

- Model \( s-(3,2) \) (Lewiński [10])

\[
\left[ h^2 \Delta - s-(3,2) S_1^2 \right] \left[ h^2 \Delta - s-(3,2) S_2^2 \right] \Psi^{(\alpha)} = 0 , \tag{39a}
\]

\[
s-(3,2) S_\alpha := 2 \sqrt{\left( 14 + (-1)^{\alpha} \sqrt{133} \right) k_G} . \tag{39b}
\]

These differential equations are known as shear equations (Cheng [15]), a very right name, as \( k_G \) is the only material parameter involved.
5.2 Solution for $\Phi^{(\xi)}$ and $\varphi^{(\xi)}$

- Model $s-(1,0)$ (Reissner [14], Panc [16])

$$D\Delta \Phi^{(1)} = -\left[ 1 + \frac{\nu_3}{10(1-\nu)k_E} h^2 \Delta \right] \sigma^{(1)} , \quad (40a)$$

$$D\Delta^2 \varphi^{(1)}_3 = \left[ 1 - \frac{2k_E - 3k_G}{10(1-\nu)k_Ek_G} h^2 \Delta \right] \sigma^{(1)} , \quad (40b)$$

- Model $s-(3,2)$

$$D\Delta \left[ h^2 \Delta - s^{-(3,2)} P^2_1 \right] \left[ h^2 \Delta - s^{-(3,2)} P^2_2 \right] \Phi^{(\alpha)} = g_\alpha \sigma^{(1)} , \quad (41a)$$

$$D\Delta^2 \left[ h^2 \Delta - s^{-(3,2)} P^2_1 \right] \left[ h^2 \Delta - s^{-(3,2)} P^2_2 \right] \varphi^{(\alpha)}_3 = h_\alpha \sigma^{(1)} . \quad (41b)$$

For an isotropic material (for a transversely isotropic material and the definitions of $g_\alpha$ and $h_\alpha$ see Westphal Jr. [9])

$$s^{-(3,2)} P^2_1 := \frac{36(1-\nu^2)}{126-121\nu^2} \left( 154 + i\sqrt{77 \frac{322 - 297\nu^2}{1-\nu^2}} \right) , \quad (42)$$

where $s^{-(3,2)} P_2$ is the complex conjugate of $s^{-(3,2)} P_1$.

5.3 Comparison with the 3-D elasticity theory

We have to compare the coefficients $s^{-(3,2)} S_\alpha$ in eqns (39) with the coefficients $S_\alpha := (2\alpha - 1)\pi$ of the 3-D elasticity, see e.g. Schwab & Wright [7]. We give these coefficients and the percentual errors for the models $s-(3,2)$ and $d-(3,2)$ (Chen & Archer [17]) in Table 2.

Table 2: Percentual errors of shear coefficients for models of twelfth-order.

<table>
<thead>
<tr>
<th>3-D Elasticity</th>
<th>$S_1 = \pi$</th>
<th>$S_2 = 3\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Errors of model $s-(3,2)$</td>
<td>0.00074%</td>
<td>7.23%</td>
</tr>
<tr>
<td>Errors of model $d-(3,2)$</td>
<td>0.028%</td>
<td>38.39%</td>
</tr>
</tbody>
</table>

The 3-D elasticity coefficients related to eqn (42) are the Papkovich-Fadle eigenvalues, see Schwab & Wright [7]. As can be seen in Figure 2, the coefficients of the model $s-(3,2)$ are close to those of the 3-D elasticity theory for all values $0 \leq \nu \leq 1/2$. 
6 The general scalar fundamental solution

We close this paper with the general scalar fundamental solution $G(r)$ of the model s-(3,2). For this purpose we make use of Hörmander's method. The RHS of eqn (28) becomes $-\delta(P,Q)m^*_t(P)$ and $u^*_j(Q) := U^*_j,(P,Q)m^*_t(P)$. With $U^*_j,(P,Q) := L^0_{r,j}G(P,Q)$ it results

$$\det(L)G(P,Q) = -\delta(P,Q) \tag{43}$$

and the determinate of $L_{r,j}$ is found to be

$$\det[L] = A_6 \left(h^2\Delta\right)^2 \left[h^2\Delta - s^{-(3,2)}S^2_1\right] \left[h^2\Delta - s^{-(3,2)}S^2_2\right]$$

$$\times \left[h^2\Delta - s^{-(3,2)}P^2_1\right] \left[h^2\Delta - s^{-(3,2)}P^2_2\right] \tag{44}$$

with

$$A_6 := \frac{9282994875(1 + \nu)k_E}{8[63(1 - \nu) - 121\nu^2] s^{-(3,2)}P^2_1 s^{-(3,2)}P^2_2} \left(\frac{1 - \nu}{2}\right)^6 \left(\frac{D}{h^2}\right)^6 \lambda^4. \tag{45}$$

We have in eqn (44) the biharmonic, the shear, and the Papkovich-Fadle equations previously calculated, see eqns (39) and (41). This shows that, as expected, the above presented advantages of the model s-(3,2) are reflected in the fundamental solution of the problem.

The general fundamental solution $G(P,Q)$ of eqn (43) is, $r := |P - Q|$

$$G_6(r) := C_1r^2 \ln r + C_2r^2 + C_3r^2 + C_4 + \sum_{i=1}^{4} C_{(4+i)}K_0(z_i)$$

$$+ \sum_{i=1}^{4} C_{(8+i)}I_0(z_i), \tag{46}$$
where

\[ z_i = \lambda_i r, \quad \lambda_\alpha := s^{-3,2} S_\alpha / h, \quad \lambda_{\alpha+2} := s^{-3,2} P_\alpha / h, \]  

(47a)

\[ C_1 = \frac{-\lambda_2^3}{4 \lambda_2^3 \lambda_2^2 \lambda_2^4} \prod_{i=1}^{3} [\lambda_4^2 - \lambda_7^2] C_8, \]  

(47b)

\[ C_2 = \frac{\lambda_4^2 \lambda_7^3 \lambda_9^2 + \lambda_2^3 \lambda_8^3 \lambda_1^2 + \lambda_4^3 \lambda_6^3 \lambda_7^2 + \lambda_4^2 \lambda_8^3 \lambda_9^2}{\lambda_1^2 \lambda_2^2 \lambda_3^2} \prod_{i=1}^{3} [\lambda_4^2 - \lambda_7^2] C_8, \]  

(47c)

\[ C_5 = -\frac{\lambda_4^4}{\lambda_1^4 [\lambda_4^2 - \lambda_7^2]} \frac{[\lambda_4^2 - \lambda_3^2]}{[\lambda_1^2 - \lambda_3^2]} C_8, \]  

(47d)

\[ C_6 = -\frac{\lambda_4^3}{\lambda_2^2 \lambda_7^3 [\lambda_4^2 - \lambda_7^2]} \frac{[\lambda_4^2 - \lambda_3^2]}{[\lambda_1^2 - \lambda_3^2]} C_8, \]  

(47e)

\[ C_7 = -\frac{\lambda_4^3}{\lambda_3^2 \lambda_7^3 [\lambda_3^2 - \lambda_7^2]} \frac{[\lambda_4^2 - \lambda_3^2]}{[\lambda_2^2 - \lambda_7^2]} C_8, \]  

(47f)

\[ C_8 = \frac{1}{2 \pi \lambda_4 h^4 \lambda_4^3 \prod_{i=1}^{3} [\lambda_4^2 - \lambda_7^2]}, \quad C_9 = C_{10} = C_{11} = C_{12} = 0. \]  

(47g)

There are two free coefficients, \( C_3 \) and \( C_4 \), in the same way as occurs with the general fundamental solution of the model \( s-(1,0) \), see Westphal Jr. et al. [18], that are expressed in the form \( (F_1 \) and \( F_2 \) are free coefficients)

\[ C_3 = F_1 C_8, \quad C_4 = F_2 C_8. \]  

(48)

7 Conclusions

The formulation here developed shows that boundary layer effects can be effectively represented in a hierarchical way. This opens up the possibility for the modeling of plate bending problems by boundary elements, incorporating a unified formulation for a hierarchy of three classes of plate models: a) the fourth-order Kirchhoff, b) the sixth-order Reissner and Mindlin, and c) the twelfth-order Poniatovskii/Reissner.

References


