The method of fundamental solutions for solving Poisson problems

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Abstract

Traditionally the method of fundamental solutions (MFS) is used to approximate solution of linear homogeneous equations. For nonhomogeneous problems, one needs to couple other numerical schemes, such as domain integration, polynomial or radial basis functions interpolation, to evaluate particular solutions. In this paper we propose to unify the MFS as a numerical method for directly approximating homogeneous solution and particular solution in a similar manner. The major advantage of such approach is that the particular solution can be easily obtained and evaluated. The numerical results show that such approach can be highly accurate.

1 Introduction

The method of fundamental solutions (MFS) was initially introduced by Kupradze and Aleksidze [11] and has been further developed by numerous mathematicians and scientists over the past 35 years. However, the method was essentially restricted to solving homogeneous elliptic equations, such as the Laplace and the biharmonic equations [7, 8]. Since 1993 Golberg and Chen [8] have extend the MFS using radial basis functions (RBFs) to solving linear and non-linear Poisson problems, inhomogenous Helmholtz equations and time dependent PDEs. The key concept of such extension is
based on the evaluation of particular solution by using the dual reciprocity method (DRM) [14] and radial basis functions (RBFs) [8]. The MFS and DRM approach have achieved mesh-free method and becomes very popular in the BEM community. The main difficulty of such scheme is the need of deriving an approximate analytical particular solutions which often become very complicated to evaluate [12].

The primary goal of this paper is to directly extend the MFS to solving nonhomogeneous problems without using the RBFs so that an easy evaluation of particular solution can be achieved. This is apparently the first time that the MFS is used directly (without RBF's) to solve a nonhomogeneous PDE. As in [1], in this paper, we will focus on solving Poisson’s equation.

Many numerical schemes for solving partial differential equations (PDEs) rely on effective methods of approximation techniques. In this paper we develop an interpolation scheme using the MFS. As a result, the MFS can be applied to interpolate functions and solve homogeneous PDEs simultaneously. Combining these two special features, a nonhomogeneous PDEs can be solved without the need of using radial basis functions or other interpolation schemes.

2 The MFS for Laplace equation

Consider the Laplace equation with Dirichlet boundary condition

$$\Delta v(P) = 0, \quad F \in \Omega,$$

$$v(P) = g(P), \quad F \in \partial \Omega \quad (2)$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. The basic idea of the MFS is to approximate the solution $v$ of equations (1)-(2) by $v_M$ expressed as a linear combination of fundamental solutions [3]

$$v_M(P) = \sum_{j=1}^{M} a_k G(P, Q_j) + a_0, \quad F \in \Omega, \quad (3)$$

where $G(P, Q) = -\log ||P - Q|| / 2\pi$ is the fundamental solution of the operator $\Delta$ and $\{Q_j\}_{j=1}^{M} = \{(x_j, y_j)\}_{j=1}^{M}$ are $M$ distinct points on the fictitious boundary $\tilde{D}$ of $\Omega$ as shown in Figure 1 for the 2D case. In general, the points $\{Q_j\}_{j=1}^{M}$ are chosen a priori in a certain fixed setting. Further details can be explained by the approximation analyses given by Bogomolny [3] and Cheng’s convergence results for Laplace’s equation with Dirichlet boundary condition when $D$ and $\tilde{D}$ are concentric circles [5]. In the numerical implementation, we have proven in our computations that the choice of placing the source points $\{Q_j\}_{j=1}^{M}$ equally spaced around a circle of radius $R$ in $\mathbb{R}^2$ or equally spaced in the polar coordinates $(\phi, \theta)$ on a sphere of radius $R$ in $\mathbb{R}^3$ gives excellent numerical results.
Once the source points have been chosen and fixed, \( \{a_j\}_1^M \cup \{a_0\} \) in (3) are normally obtained from simple collocation. By choosing \( M + 1 \) points \( \{P_k\}_1^{M+1} \) on \( D \) and \( M \) points \( \{Q_j\}_1^M \) on the fictitious boundary \( \hat{D} \) that satisfy the equations (for Dirichlet boundary condition), we obtain

\[
\sum_{j=1}^{M} a_j G(P_k, Q_j) + a_0 = g(P_k), \quad 1 \leq k \leq M + 1.
\]  

From equation (4) it can be observed that the system will become highly ill-conditioned as the value of \( R \) increases [4, 7, 15]. It is natural to put a limit on the value of \( R \) to approximately \( 5 \sim 10 \) times of the diameter of \( D \). Despite the ill-conditioning problem, the accuracy of the numerical solution is generally insensitive to the large value of \( R \) [8].

### 3 Surface interpolation using the MFS

To extend the MFS to nonhomogeneous problems, we consider the following Poisson’s equation

\[
\Delta u(P) = f(P), \quad P \in \Omega, \tag{5}
\]
\[
u(P) = g(P), \quad P \in \partial \Omega. \tag{6}
\]

Let \( u_p \) be a particular solution of (5)

\[
\Delta u_p(P) = f(P) \tag{7}
\]

but does not necessarily satisfy the boundary condition (6). Let \( v = u - u_p \). Then, \( v \) satisfies

\[
\Delta v(P) = 0, \quad P \in \Omega, \tag{8}
\]
\[
v(P) = g(P) - u_p(P), \quad P \in \partial \Omega. \tag{9}
\]

Notice that the approximate solution \( \hat{v} \) of \( v \) can be obtained as described in the last section provided that \( u_p \) can be successfully evaluated. Since
$u_p$ is not unique, it allows a wide variety of choices for finding a suitable particular solution. In fact, in recent years, the evaluation of approximate particular solution of $u_p$ is a subject of intensive research. Traditionally $u_p$ was evaluated by Newtonian potential

$$u_p(P) = \int_{\Omega} f(P) G(P, Q) d\nu$$

(10)

which required tedious domain integration. In his paper, Atkinson [2] gave three different ways to evaluate $u_p$ in (7). A review for the particular solution was given by Golberg [10]. In the engineering literature, the DRM [14] and the MRM [13] are mostly widely used to directly approximate the particular solution. In particular, RBFs have been widely used in this respect in the DRM literature. One of the most popular choices of the RBFs is the thin plate splines (TPS) [9]

$$\varphi(r) = \begin{cases} r^2 \log r, & \text{in } \mathbb{R}^2, \\ r, & \text{in } \mathbb{R}^3. \end{cases}$$

(11)

In the DRM, we approximate $f(P)$ in (7) by a linear combination of basis function $\{ \varphi_j \}_{j=1}^n$; i.e.,

$$f(P) \simeq \hat{f}(P) = \sum_{j=1}^{n} a_j \varphi_j(P)$$

(12)

where $\{a_j\}_{j=1}^n$ are undetermined coefficients. By collocation,

$$f(P_k) = \hat{f}(P_k) = \sum_{j=1}^{n} a_j \varphi_j(P_k), \quad 1 \leq k \leq n,$$

(13)

where $\{P_k\}_{k=1}^n$ are $n$ points in $\mathbb{R}^d$. Least-squares method can also be used if the number of collocation and source points are not equal. Notice that since the particular solution does not have to satisfy the boundary condition, the interpolation points $\{P_k\}_{k=1}^n$ can be lied outside the domain. For the traditional DRM, $\{P_k\}_{k=1}^n$ are chosen inside the domain. Assuming that (13) can be solved uniquely for $\{a_j\}_{j=1}^n$, the approximate particular solution $\hat{u}_p$ is given by

$$\hat{u}_p(P) = \sum_{j=1}^{n} a_j \Phi_j(P_k)$$

(14)

where

$$\Delta \Phi_j = \varphi_j, \quad 1 \leq j \leq n.$$ 

(15)

To achieve high efficiency, it is important to solve (15) analytically. For Laplacian $\Delta$, the closed-form $\Phi_j$ is not difficult to obtain. However, for other differential operators, the derivation of $\Phi_j$ is not trivial. A great deal of
effort in deriving the closed-form particular solution for various differential operators \[8\].

We have observed that the TPS \( \varphi(r) \) in (11) is a fundamental solution of the biharmonic equation

\[
\Delta^2 \varphi = -\delta,
\]

(16)

where \( \delta \) is the Dirac delta. Motivate by this observation, we seek an approximation based on fundamental solution \( \varphi_\lambda \) of the following eigenvalue equation, which is also a modified Helmholtz equation,

\[
(\Delta - \lambda) \varphi_\lambda = -\delta
\]

(17)

where \( \lambda \in \mathbb{R}^+ \setminus \{0\} \) and

\[
\varphi_\lambda = \begin{cases} 
\frac{1}{2\pi} K_0(\sqrt{\lambda} r) & \text{in } \mathbb{R}^2, \\
\frac{1}{4\pi r} e^{\sqrt{\lambda} r} & \text{in } \mathbb{R}^3.
\end{cases}
\]

(18)

In particular,

\[
\Delta \varphi_\lambda(P) = \lambda \varphi_\lambda(P), \quad P \in \mathbb{R}^d \setminus \{0\}.
\]

(19)

Similar to the TPS interpolation in (12), an approximation of \( f \) can be written as the linear combination of fundamental solutions which contains different wave lengths; i.e.,

\[
\hat{f}(P) = \sum_{i=1}^{q} \sum_{k=1}^{n} \alpha_{ik} \varphi_{\lambda_i, k},
\]

(20)

where \( q \) is the number of wave length. Note that \( \varphi_\lambda \) has a singularity at \( r = 0 \). To avoid the singularity using interpolation scheme, we choose the source points outside the domain as shown in Figure 2. One major advantage of choosing \( \{\varphi_{\lambda_i}\} \) as basis function is that the particular solution can be obtained directly. It is now clear that if the approximate particular solution \( \hat{u}_p \) is written in the form

\[
\hat{u}_p(P) = \sum_{i=1}^{q} \sum_{k=1}^{n} \beta_{ik} \varphi_{\lambda_i, k},
\]

(21)

then by

\[
\Delta \hat{u}_p = \sum_{i=1}^{q} \sum_{k=1}^{n} \beta_{ik} \Delta \varphi_{\lambda_i, k} = \sum_{i=1}^{q} \sum_{k=1}^{n} \beta_{ik} \lambda_i \varphi_{\lambda_i, k}.
\]

(22)

This implies that if we choose \( \beta_{ik} = \alpha_{ik}/\lambda_i \) in (19), then \( \hat{u}_p \) in (20) is indeed an approximate particular solution. In contrast to the RBF approach, it is interest to observe that the basis functions of particular solution in (21) and interpolation in (20) are the same. In numerical implementation, we use
least-squares method instead of collocation method. More collocation points in the domain than source points on the fictitious boundary are obviously needed.

Figure 2. The MFS for surface interpolation.

To justify the approximation of \( f \) in (20), we need the following theorems [1].

**Theorem 1** Let \( \{Q\}_1 \notin \Omega \). The functions

\[
    \varphi_{\lambda_1}(P, Q_1), \ldots, \varphi_{\lambda_1}(P, Q_n); \ldots, \varphi_{\lambda_\rho}(P, Q_1), \ldots, \varphi_{\lambda_\rho}(P, Q_n)
\]

restricted to any open set \( \Omega \) are linear independent.

**Theorem 2** Let \( \tilde{\Gamma} \) be an admissible source set and \( I \) an interval in \( (-\infty, 0] \). The space

\[
    S_{\tilde{\Gamma}, I, \Omega} = \text{Span}\{\varphi_{\lambda}(P, Q) : Q \in \tilde{\Gamma}, \lambda \in I\}
\]

is dense in \( L^2(\Omega) \).

### 4 Numerical results

To demonstrate the effectiveness of our proposed approach, we consider a benchmark example [6] for Poisson’s equation (5)-(6) where

\[
f(x, y) = \frac{751\pi^2}{144} \sin \frac{\pi x}{6} \sin \frac{7\pi x}{4} \sin \frac{3\pi y}{4} \sin \frac{5\pi y}{4} + \frac{7\pi^2}{12} \cos \frac{\pi x}{6} \cos \frac{7\pi x}{4} \sin \frac{3\pi y}{4} \sin \frac{5\pi y}{4},
\]

\[
g(x, y) = \sin \frac{\pi x}{6} \sin \frac{7\pi x}{4} \sin \frac{3\pi y}{4} \sin \frac{5\pi y}{4},
\]

and \( \Omega \cup \partial \Omega = [1, 2] \times [1, 2] \). The graph of nonhomogeneous term \( f(x, y) \) is shown in Figure 3. The exact solution \( u(x, y) \) is given by

\[
u(x, y) = \sin \frac{\pi x}{6} \sin \frac{7\pi x}{4} \sin \frac{3\pi y}{4} \sin \frac{5\pi y}{4}.
\]
To interpolate the forcing term \( f(x, y) \), we choose 400 quasi-random interior points and 40 uniformly distributed points on the fictitious boundary which is a circle with center at \((1.5, 1.5)\) and radius 5. The wave numbers in (20) are choosing to be \(\{1, 10, 20, \ldots, 80\}\). Least square method has been used for this purpose. We use least square subroutine DLSQRR from IMSL library.

To evaluate the approximate homogeneous solution \( \hat{v} \) of (8)-(9), we choose 40 uniformly distributed points on the physical boundary and the same number of points on the fictitious boundary same as above. Collocation method has been used to evaluate \( \hat{v} \). We perform our numerical test on 400 quasi-random points in the domain. The maximum absolute error of \( u \) is \(7.452 \times 10^{-9}\). The maximum absolute interpolation error of \( f \) is \(1.228 \times 10^{-10}\). The overall profile of absolute error of \( u \) and \( f \) are shown in Figures 4 and 5.

Table 1. The effect of \( u \) and \( f \) for various number of wave lengths.

<table>
<thead>
<tr>
<th># of wave length</th>
<th>(|u - \hat{u}|_\infty)</th>
<th>(|f - \hat{f}|_\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.855</td>
<td>0.948</td>
</tr>
<tr>
<td>5</td>
<td>(7.673E - 2)</td>
<td>(4.865E - 2)</td>
</tr>
<tr>
<td>6</td>
<td>(8.927E - 3)</td>
<td>(9.698E - 4)</td>
</tr>
<tr>
<td>7</td>
<td>(5.431E - 4)</td>
<td>(1.097E - 5)</td>
</tr>
<tr>
<td>8</td>
<td>(6.931E - 6)</td>
<td>(7.100E - 8)</td>
</tr>
<tr>
<td>9</td>
<td>(7.452E - 9)</td>
<td>(1.228E - 10)</td>
</tr>
</tbody>
</table>

In Table 1, we show the maximum absolute errors of \( u \) and \( f \) for various wave lengths with 40 fixed source points on the fictitious boundary. This reveals that the number of wave length has great impact on the numerical solution of \( u \). Double precision is used for all the numerical computation.
5 Conclusions

In the past the MFS has been used to solve the homogeneous equation.
In this paper we propose to use the MFS to interpolate the forcing term
directly. As a result, the particular solution of the nonhomogeneous solution can be evaluated easily. The preliminary numerical results shows that our approach is robust and compare favorable to the RBF approach. There are still a number of issues such as the number of wave numbers, the number and the location of source points needed to be further studied. The extension of this work to more general differential equations such as nonlinear and time-dependent problems is currently under investigation.

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References


