Boundary meshfree methods based on the boundary point interpolation methods

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Abstract

A group of meshfree method based on Boundary Integral Equation (BIE) have been proposed and developed in order to overcome drawbacks in the conversional Boundary Element Method (BEM) that require boundary elements in constructing shape functions. In this paper, two boundary meshfree methods that use point interpolation methods (PIM), the Boundary Point Interpolation Method (BPIM) using the polynomial PIM and the Boundary Radial Point Interpolation Method (BRPIM) using the radial PIM, are reviewed and assessed. The numerical implementations of these two methods are examined and compared with each other on several technical issues in great details, including the size of the support domain, the convergence, the performance, and so on. These two boundary-type meshfree methods are also compared with the Boundary Node Method (BNM) and the conversional BEM in both efficiency and performance. Several numerical examples of 2-D elastostatics are analyzed using BPIM and BRPIM. It is found that the BPIM and BRPIM are very easy to implement, and very robust for obtaining numerical solutions for problems of computational mechanics. Key issues related the future development of boundary meshfree methods are also discussed.

1 Introduction

Mesh free (or meshless) method has attracted more and more attention from researchers in recent years, and it is regarded as a potential new generation of numerical methods in computational mechanics. A mesh free does not require a mesh to discretize the problem domain, and the approximate solution is
constructed entirely based on a set of scattered nodes. Several ‘domain’ type
meshless methods, such as Element Free Galerkin (EFG) method, Reproducing
Kernel Particle Method (RKPM), Point Interpolation Method (PIM) [1], Point
Assembly Method (PAM)[2], Meshless Local Petrov-Galerkin (MLPG) method
have been proposed and achieved remarkable progress in solving a wide range of
static and dynamic problems for solids and structures. Techniques of coupling
meshless methods with other established numerical methods have also been
proposed, such as coupled EFG/Boundary Element Method (BEM) [3].

The Boundary Element Method (BEM) is a numerical technique based on
Boundary Integral Equation (BIE), which has been developed since 1960’s. For
many problems, BEM is undoubtedly superior to the ‘domain’ type methods,
such as FEM. BEM has a well-known dimensionality advantage for linear
problems. For example, only 2-D bounding surface of a 3-D body needs to be
discretized. However, in BEM, meshing is also a burdensome and expensive task
for some problems, such as complicated boundary problems, 3-D problems and
moving boundary problems.

Therefore, the idea of meshless has also been used in BIE. The Moving Least
Squares (MLS) approximation is combined with BIE to propose a boundary type
meshless method called Boundary Node Method (BNM) [4]. However, because
the MLS shape functions lack Kronecker delta function properties, it is difficult
to accurately satisfy the boundary condition in BNM. This problem becomes
even more serious in the boundary type meshless method because a large number
of boundary conditions need be satisfied. The method used in BNM to impose
boundary conditions doubles the number of system equations, making BNM
computationally much more expensive than the conventional BEM.

Two point interpolation techniques, Point Interpolation Methods (PIM)[1],
which use polynomial and radial functions as basis respectively, have been
proposed to construct meshfree shape function with Kronecker delta function
properties. The polynomial PIM possesses some distinguished advantages.

- The PIM shape functions possess Kronecker delta function properties.
  Therefore, the boundary conditions can be easily enforced.
- Theoretically, the integrations of shape functions and its derivatives can
  be accurately obtained, because shape functions so-formed are
  polynomials.

However, the only issue on its way is the possible singularity of the moment
matrix. It is tricky in choosing polynomial basis according the nodal distribution
in the influence domain. If an inappropriate polynomial basis is chosen, it may
result in a badly conditioned moment matrix, which could be even non-invertible.
Some strategies have been developed for alleviating this problem. An efficient
Matrix Triangularization Algorithm (MTA) is proposed and developed recently
[5] to avoid the interpolation singularity of polynomial PIM. In addition, using
radial functions as basis in PIM can also avoid the interpolation singularity.

PIMs have been combined with BIE to proposed two boundary type meshfree
methods, the boundary point interpolation method (BPIM)[6] and the boundary
radial point interpolation method (BRPIM)[7]. In BPIM and BRPIM, the
boundary of problem domain is discretized by properly scattered nodes. The BIE
for 2-D elastostatics is discretized using PIMs. Because PIM shape functions possess delta function properties, BPIM and BRPIM overcome the shortcomings of BNM. The imposition of boundary conditions in BPIM and BRPIM is as easy as in the traditional BEM. BPIM and BRPIM possess the same number of system equations as the conventional BEM. In addition, the rigid body movement can also be utilized to avoid some singular integrals.

In this paper, BPIM and BRPIM are reviewed and assessed in great details. The numerical implementations of these two methods are examined and compared with each other on several technical issues in great details, including the size of the support domain, the convergence, the performance, and so on. These two meshfree methods are also compared with BNM and the conventional BEM. Numerical examples of 2-D elastostatics are analyzed using these two boundary meshfree methods.

2 Point interpolation methods

2.1 Polynomial basis point interpolation

Consider a function \( u(x) \) defined in domain \( \Omega \) discretized by a set of field nodes. The point interpolates \( u(x) \) from the surrounding nodes of a point \( x \) using the polynomials as basis can be written as

\[
\begin{align*}
\hat{u}(x) &= \sum_{i=1}^{n} p_i(x) a_i = \mathbf{p}^T(x) \mathbf{a} \\
\end{align*}
\]  

(1)

where \( p_i(x) \) is a monomial in the space coordinates \( x^T = [x, y] \), \( n \) is the number of nodes in the neighborhood of \( x \), \( a_i \) is the coefficient for \( p_i(x) \) corresponding to the given point \( x \). The \( p_i(x) \) in equation (1) is built utilizing Pascal’s triangle, so that the basis is complete.

The coefficients \( a_i \) in equation (1) can be determined by enforcing equation (1) at the \( n \) nodes surrounding point \( x \), it can be written in the following matrix form:

\[
\begin{align*}
\mathbf{u}^e &= \mathbf{P}_o \mathbf{a} \\
\end{align*}
\]  

(2)

where

\[
\begin{align*}
\mathbf{u}^e &= [u_1, u_2, u_3, \ldots, u_n]^T \\
\mathbf{P}_o^T &= [p(x_1), p(x_2), p(x_3), \ldots, p(x_n)] \\
\end{align*}
\]  

(3)

(4)

From equation (2), we have

\[
\mathbf{a} = \mathbf{P}_o^{-1} \mathbf{u}^e
\]

(5)

Hence,

\[
\begin{align*}
\hat{u}(x) &= \phi(x) \mathbf{u}^e \\
\end{align*}
\]  

(6)

where the shape function \( \phi(x) \) is defined by

\[
\phi(x) = \mathbf{p}^T(x) \mathbf{P}_o^{-1} = [\phi_1(x), \phi_2(x), \phi_3(x), \ldots, \phi_n(x)]
\]

(7)

It can be found that PIM shape functions possess the delta function property.

The polynomial PIM is accurate and easy to use. However, like other methods that use polynomial as basis functions, it is tricky in choosing basis for interpolation in polynomial PIM. If an inappropriate polynomial basis is chosen,
it may result in a badly conditioned matrix, which could be even non-invertible, due to rank deficiency of basis functions.

In order to avoid the singularity of matrix \( P_0 \), several strategies have been proposed. Liu and Gu[1] proposed a moving node method to slightly change the coordinates of nodes randomly before computation. Changing basis function through the transformation of the local coordinate is the other effective method. Recently, the MTA[5] has been proposed that is an automatic procedure to ensure a proper node enclosure and a proper basis selection. The moment matrix is triangularized to obtain their row and column ranks. In the triangularization process, we can also obtain the information about which nodes need to be excluded from the influence domain and which monomials need to be removed from the basis. Using MTA can ensure a successful construction of PIM shape functions.

### 2.2 Radial basis point interpolation

The point interpolation form (1) is re-written as:

\[
\begin{align*}
\mathbf{u}(\mathbf{x}) &= \sum_{i=1}^{n} R_i(r) a_i + \sum_{j=1}^{m} p_j(x) b_j, \\
\mathbf{a} &\in \mathbb{R}^{n}, \quad \mathbf{b} &\in \mathbb{R}^{m}
\end{align*}
\]

(8)

with the constraint condition

\[
\sum_{i=1}^{n} p_j(x) a_i = 0, \quad j=1-m
\]

(9)

where \( R_i(r) \) is the radial basis functions, \( n \) is the number of nodes in the neighborhood of \( x \), \( p_j(x) \) is monomials in the space coordinates, \( m \) is the number of polynomial basis functions, coefficients \( a_i \) and \( b_j \) are interpolation constants. In the radial basis function \( R_i(r) \), the variable is only the distance, \( r \), between the interpolation point \( x \) and a node \( x_i \). The Multi-quadrics (MQ) radial function is used in this paper for simplification. Two parameters (\( q \) and \( C \)) need be determined for each radial basis function.

The second term of equation (8) consists of polynomials. It does not have to be here, but it has been proved that it can improve interpolation accuracy. In practice, only a limited number of \( p_j(x) \) is sufficient, i.e. \( m \ll n \). We often use only the linear polynomial term.

Coefficients \( a_i \) and \( b_j \) in equation (8) can be determined by enforcing equation (8) to be satisfied at the \( n \) nodes surrounding point \( x \). Equation (8) can be re-written in matrix form as follows

\[
\begin{bmatrix}
\mathbf{u} \\
\mathbf{0}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{R}_0 & \mathbf{P} \\
\mathbf{P}^T & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{a} \\
\mathbf{b}
\end{bmatrix} = \mathbf{G} \mathbf{a}_0
\]

(10a)

where

\[
\mathbf{P}^T =
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
y_1 & y_2 & \cdots & y_n
\end{bmatrix}
\]

\[
\mathbf{R}_0 =
\begin{bmatrix}
R_1(r_1) & R_2(r_1) & \cdots & R_n(r_1) \\
R_1(r_2) & R_2(r_2) & \cdots & R_n(r_2) \\
\vdots & \vdots & \ddots & \vdots \\
R_1(r_n) & R_2(r_n) & \cdots & R_n(r_n)
\end{bmatrix}
\]

(10b)
Because the matrix $R_0$ is symmetric, the matrix $G$ will also be symmetric. Equation (10a) can be solved in the following procedure. From equation (8), we have

$$a = R_0^T u_e - R_0^T P_m b$$  
(11)

Substituting of the above expression into equation (9) gives

$$b = S_b u_e, \quad S_b = [P_m^T R_0^T P_m]^{-1} P_m^T R_0^{-1}$$  
(12)

Substituting equation (12) back into equation (11), we obtain

$$a = S_a u_e, \quad S_a = R_0^{-1} [1 - P_m S_b]$$  
(13)

The interpolant equation (8) finally expressed as

$$u(x) = [R^T(x) S_a + p^T(x) S_b] u_e = \Phi(x) u_e$$  
(14a)

where the shape function $\Phi(x)$ is defined by

$$\Phi(x) = [\phi_1(x), \phi_2(x), \ldots, \phi_n(x)] = R^T(x) S_a + p^T(x) S_b$$  
(14b)

Mathematicians have proved the existence of $R_0^{-1}$.

### 2.3 Boundary point interpolation techniques

The boundary point interpolants in BPIM and BRPIM are constructed on the 1-D bounding curve $\Gamma$ of a 2-D domain $\Omega$, using a set of discrete nodes on $\Gamma$. As in the conventional BEM formulation, the displacement and traction of boundary nodes can be constructed independently using polynomial PIM equation (6) and radial PIM equation (14a). The PIM formulations for displacement $u(s)$ and traction $t(s)$ at a boundary point $s$ on the boundary $\Gamma$ from the surrounding boundary nodes uses PIMs

$$u(s) = \Phi^T(s) u_n, \quad t(s) = \Phi^T(s) t_n$$  
(15)

where the shape function $\Phi(s)$ is defined for polynomial PIM and radial PIM.

### 3 BPIM and BRPIM formulations

#### 3.1 Discrete equations of BPIM

The well-known BIE formulation for 2-D linear elastostatics is given by

$$c_i u_j + \int_{\Gamma} u t^* d\Gamma = \int_{\Gamma} u t d\Gamma + \int_{\Omega} b u^* d\Omega$$  
(16)

where $c_i$ is a coefficient depended on the geometrical shape of boundary. $b$ is the body force vector, $u^*$ and $t^*$ are the fundamental solution for linear elastostatics. Substituting equation (15) into equation (16) yields the BPIM and BRPIM system equations

$$H u_n = G t_n + d$$  
(17)
where \[ H = \mathbf{c} + \int_{\Gamma} ^{i} \mathbf{t} \mathbf{\Phi}^T d\Gamma, \quad G = \int_{\Gamma} ^{i} \mathbf{u} \mathbf{\Phi}^T d\Gamma, \quad d = \int_{\Omega} ^{} \mathbf{b} \mathbf{u}^T d\Omega \] (18)

There are two types of boundary conditions in BPIM and BRPIM

\[ t = \bar{t} \quad \text{on the natural boundary } \Gamma^t \] (19)

\[ u = \bar{u} \quad \text{on the essential boundary } \Gamma^u \] (20)

Because the shape functions of BPIM and BRPIM have the delta properties, the boundary conditions can be imposed in the same way as the traditional BEM. After applying the boundary conditions, the system equation (17) has \( 2N_B \) equations and \( 2N_B \) unknowns for \( N_B \) boundary nodes.

3.2 Comparison between BPIM, BRPIM, BNM and BEM

A comparison between boundary point interpolation methods (BPIM, BRPIM), BNM and BEM is summarized here. It can be found that BPIM (BRPIM), BNM and BEM are all based on the boundary integral equation. The difference is in the means of implementation. For simplification, these two boundary point interpolation methods are all called the BPIM in this section.

a) BPIM versus BEM

Both BPIM and BEM use the point interpolants, in which the number of monomials used in the base functions, \( m \), is the same as the number of nodes, \( n \), utilized. Therefore, the interpolation functions possess the Kronecker delta function properties. The boundary conditions can be implemented with ease.

However, the BPIM is a boundary type meshfree method when the BEM is a boundary type numerical method based on the mesh. As other meshfree methods (e.g. EFG, BNM, MLPG), the interpolation procedure in the BPIM is based only on a group of arbitrary distributed nodes as above discussed. The interpolation procedure in the BEM is based on an element. The interpolation at a sampling point in the BPIM is performed over the support domain of the point, which may overlap with the support domains of other sampling points. BEM defines the shape functions over pre-defined regions called elements, and there is no overlapping at all.

b) BPIM versus BNM

Both BPIM and BNM are boundary type meshless methods. The difference between these two methods comes from the different interpolants utilized. As above discussed, the BPIM uses passing nodes point interpolants, in which the coefficients \( a \) and \( b \) in interpolants are constant. The Moving Least Squares approximation are used in the BNM, in which \( a(s) \) and \( b(s) \) are also functions of curvilinear co-ordinate \( s \). Therefore, the shape function of BNM is more complicated than that of BPIM. In addition, the shape function of BNM constructed using the MLS approximation lacks the delta function property. It takes extra effort to impose boundary conditions. One can observe that the number of system equations BNM is double of that of the BPIM. It is because \( 2N_B \) additional equations have to be used to enforce the boundary conditions. It leads the computational cost of the BNM more expensive than that of the BPIM.
4 Numerical implementation of BPIM and BRPIM

4.1 Singular integral

In order to obtain the integrals in equations (18), background integration cells that can be independent of the nodes are required. It is can be seen that the integrands in equations (18) consist of regular and singular functions. The regular integration can be evaluated using the usual Gaussian quadrature based on the integration cells. In equation (18), the matrix $G$ contains a log singular integral. This type of singular integral can be evaluated by log Gaussian quadrature.

In matrix $H$, $c$ is a coefficient depended on the geometrical shape of the boundary, which is easy to be obtained for a smooth boundary. However, it is more complicated to obtain $c$ for non-smooth boundaries. In addition, $H$ contains $(1/r)$ type singular integral. Therefore, it can be a non-trivial task to directly evaluate the diagonal terms of $H$. Note that shape functions of BPIM and BRPIM possess the Kronecker delta function properties, therefore, the rigid body movement can be utilized in this work to obtain the diagonal terms of $H$.

4.2 Handling of corners with traction discontinuities

In handling traction discontinuities in corners, special care should be taken. Double nodes and discontinuous elements at corners are used to overcome this problem in the traditional BEM. Because there are no elements used in BPIM and BRPIM, a simple method proposed here to solve this difficulty is by displacing the nodes from the corner. In addition, the influence domain for interpolation is truncated at the corner. The method is very easy to implement and is used in the following numerical examples. The simple method is proven to be very accurate.

5 Numerical examples

The BPIM and BRPIM are applied to obtain the solution of a cantilever beam, which is shown in Figure 1. A plane stress problem is considered. The elastic constants for the beam are: $E=3.0\times10^7$, and $v=0.3$. The length $L$ and height $D$ of beam are 48 and 12, respectively. The beam is subjected to a parabolic traction at the free end. This is a benchmark problem. The analytical solution is available and can be found in a textbook by Timoshenko and Goodier.

![Figure 1: A cantilever beam](image)

![Figure 2: Nodal arrangement](image)
Parameters on the performance of the present BPIM and BRPIM are investigated first. In following parameter investigations, a total of 120 uniform boundary nodes, as shown in the Figure 2, are used to discretize the boundary of the beam. One hundred twenty uniform integration cells are used to evaluate the integral of matrixes. We define the following norm as an error indicator using the cantilever beam to reflect the accuracy.

\[
e_i = \frac{1}{N} \sqrt{\frac{\sum_{i=1}^{N} (\tau_i - \bar{\tau})^2}{\sum_{i=1}^{N} \bar{\tau}^2}}
\]

where \( N \) is the number of nodes investigated, \( \tau \) is the shear stress obtained numerically, and \( \bar{\tau} \) is the analytical shear stress.

### 5.1 Effects of radial function parameters

The Multi-quadrics (MQ) radial function is used as basis function in BRPIM. Two parameters, \( C \) and \( q \), will influence the performance of MQ. \( C \) is defined as

\[
C = c_0 d_i
\]

where, \( c_0 \) is a coefficient chosen. The \( d_i \) is the shortest distance between the node \( i \) and neighbor nodes.

The parameter, \( q \), is firstly investigated. Shear stresses for different \( q \) are obtained and compared with the analytical solution. It can be observed that \( q=0.5 \) and \( c_0=1.0-6.0 \) leads to a better result in the range of studies. Hence, \( q=0.5 \) and \( 0.5 \) and \( c_0=2.0 \) are used in following studies.

### 5.2 Effects of support domain

The size of influence domain of a quadrature point is decided by the parameter \( \alpha_s \) in following equation.

\[
r_s = \alpha_s d_i
\]

Results of \( \alpha_s=1.0-5.0 \) are obtained and plotted in Figure 3. It can be found that results of \( \alpha_s=3.0-4.5 \) (about 6-10 nodes used in a influence domain) are very good. A too small influence domain (\( \alpha_s<2.5 \)) and a too big influence domain (\( \alpha_s>4.5 \)) lead big errors. The poor accuracy of a too small influence domain is because that there are not enough nodes to perform interpolation for the field variable. In the contrary, a too big influence domain will increase the numerical error of interpolation because there are too many nodes to perform interpolation. Therefore, \( \alpha_s=3.0-4.5 \) can obtain an acceptable result. For convenience and consistency, \( \alpha_s=3.0 \) will be used in the following studies.

### 5.3 Results of the beam

Figure 4, illustrates the comparison between the shear stress calculated analytically and by the BPIM and BRPIM at the section of \( x=L/2 \). The plot shows
a good agreement between the analytical and numerical results. The conventional linear BEM results of this problem are also shown in the same figure for comparison. The density of the nodes in the BEM and BPIM is exactly the same. It is clearly shown that the BPIM and BRPIM results are more accurate than BEM results. This is because the BRPIM uses more nodes for the interpolation of the displacement and traction.

The convergence for the shear stresses at the section of \( x=L/2 \) with mesh refinement is shown in Figure 5, where \( h \) is a characteristic length equivalent to the maximum element size in the BEM. Three kinds of nodal arrangement of 72, 240 and 480 uniform boundary nodes are used. It is observed that the convergence of the BPIM and BRPIM is very good. The convergences of BNM and the conventional linear BEM are also shown in the same figure. It can also be observed that BPIM and BRPIM have higher accuracy than that of BEM and BNM.

6 Remarks

Two boundary type meshfree methods, boundary point interpolation method (BPIM) and Boundary Radial point interpolation method (BRPIM), for solving 2-D problems of elastostatics are proposed in this chapter. Some important parameters on the performance of the present method are investigated in great detail. From the studies in this chapter, the following conclusion can be drawn:

1) For MQ radial function, \( q=0.5 \) and \( c_\phi=1.0-6.0 \) leads to acceptable results for most problems studied. \( q=0.5 \) and \( c_\phi=2.0 \) are recommended in BRPIM.
2) The size of influence domain of \( \alpha_s=3.0-4.5 \) should be used for most problems studied.
3) Numerical examples are presented to demonstrate the convergence, validity and efficiency of the present methods. It is demonstrated that the BPIM and
BRPIM are easy to implement, and very flexible for solving 2-D problems of elastostatics.

The BPIM and BRPIM can be extended to solve 3-D problems to take the full advantages of the meshless concept. However, more research work need be done in utilization BPIM and BRPIM to 3-D problems.

![Figure 5: Convergence of BPIM and BRPIM](image)

**Reference**


