Determination of optimal threshold for matrix compression in wavelet BEM

K. Koro & K. Abe

Department of Civil Engineering and Architecture,
Niigata University, Japan

Abstract

This paper presents a practical determination method for optimal threshold in wavelet BEM. The optimal threshold, used for truncation of coefficient matrix entries, is determined such that truncation error is comparable to discretization error, in order to minimize the computational cost without deterioration of the accuracy. Errors are estimated in the residual sense. The discretization error is estimated based on the asymptotical behaviour of the residual \( R \). These enable us to determine the optimal threshold in advance, independently of prescribed analytical conditions. Through numerical example, availability and validity of the proposed strategy are investigated.

1 Introduction

Boundary element method using wavelet bases (wavelet BEM) is one of the fast solution for BE analysis. Since the wavelets possess a local support and vanishing moment property, coefficients of boundary element equations have higher asymptotical order with respect to a distance between source and integral points than that of conventional bases. As a result, enhancement of the computational performance is achieved by compressing the coefficient matrix of boundary element equations.

The coefficient matrices are compressed by truncation of small entries, i.e. by neglecting the entries which do not exceed a prescribed threshold. Sparseness of the matrices depends on not only order of vanishing moments
but also magnitude of the threshold. Inappropriate threshold results in either deterioration of the accuracy or decrease of sparseness. Hence, there exists the optimal threshold that leads to the least computational cost without deteriorating the accuracy, and determination of its value is necessary for the practical use of wavelet BEM.

In many studies on wavelet BEM (e.g. Goswami et al [1] and the authors [2]), the threshold has been selected empirically and validity of its determination has scarcely been discussed. The determination of the threshold has been attempted by Dahmen et al [3] and Petersdorff et al [4]. Although the justification of these strategies based on asymptotical analysis of error is guaranteed mathematically, application will be imposed restrictions, e.g. type of boundary conditions.

In the present paper, a practical determination of the optimal threshold is developed for wavelet BEM. The optimal threshold is determined such that the truncation error is comparable to the discretization error. These two kinds of errors are estimated in the residual sense, and thereby the estimation of the discretization error is carried out using the residual $R$ proposed by Abe [5]. Since the $R$ for an actual discretization cannot be evaluated a priori without solving boundary element equations, it is estimated based on asymptotical behaviour given by numerical experiments in advance. In spite of this additional analyses, since the degrees of freedom (DOFs) of them are small, the computational cost in this strategy will be negligible comparing with that of the main analysis.

2 Wavelet BEM

2.1 Non-orthogonal wavelets

Wavelets, used as the bases of approximations, consist of two kinds of functions, i.e. scaling function $\phi$ and wavelet $\psi$. In this study, we employ B-splines and non-orthogonal spline wavelets proposed by Koro & Abe [2] as $\phi$ and $\psi$. That is,

$$
\begin{align*}
\phi(\xi) &= \frac{1}{m!} \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (\xi - j)_+^m, \\
\psi(\xi) &= \frac{\alpha_n^m}{m!} \sum_{j=0}^{m+n+1} \binom{m+n+1}{j} (2\xi - j)_+^m
\end{align*}
$$

(1)

where $\alpha_n^m$ is a constant to normalize $\psi(\xi)$, and $(\cdot - j)_+^m$ represents the truncated power function of degree $m$. $n$ is the order of vanishing moments.

For wavelet series on a finite interval, boundary scaling functions $\tilde{\phi}$ and boundary wavelets $\tilde{\psi}$ are required to guarantee the completeness of wavelet expansion. $\tilde{\phi}_i \ (i = 1, \ldots, m)$ are constructed by B-spline with
multiple knots; \( \tilde{\psi}_i (i = 1, \ldots, N_b) \) are given by

\[
\tilde{\psi}_i (\xi) = \begin{cases} 
\psi_{n+1}(\xi) & (m = 0) \\
\alpha_k^{m} \left[ \phi_1(2\xi) + p_i \phi_{i+1}(2\xi) \right] & (i < m, m \geq 1, n = 1) \\
\alpha_k^{m} \left[ \phi_1(2\xi) + p_i \phi(2\xi) \right] & (i = m, m \geq 1, n = 1) \\
\alpha_k^{m} \left[ \tilde{\psi}_{1(n-1)}(\xi) + a_{i,n-1} \psi_{n-1}(\xi) \right] & (i \leq m, m \geq 1, n \geq 2) 
\end{cases}
\] (2)

where \( \alpha_k^{m} \) is a constant to normalize \( \tilde{\psi}_i \). \( p_i \) and \( a_{i,n-1} \) are determined so that \( \tilde{\psi}_i \) may have vanishing moments with a prescribed order. When these bases are used, \( m \) and \( n \) have to be selected such as \( m + n = (\text{odd}) \).

2.2 Boundary element equations

Let us consider two-dimensional Laplace problems. A boundary integral equation is derived by a direct method as follows:

\[
c(x)u(x) + \int_{\Gamma} q^*(x, y)u(y) \, d\Gamma_y - \int_{\Gamma} u^*(x, y)q(y) \, d\Gamma_y = 0, \quad (3)
\]

where \( u \) and \( u^* \) are the potential and its fundamental solution, respectively, and \( q \) and \( q^* \) are given by their outward normal derivatives. \( \Gamma \) is a boundary, and \( x, y \in \Gamma \). \( c(x) \) is a free term.

To obtain boundary element equations, we introduce wavelet series \( \tilde{u} \) and \( \tilde{q} \) as approximations of \( u \) and \( q \), i.e.

\[
\tilde{u} = \sum_{i=1}^{N} \hat{U}_i w_i = \sum_{i=1}^{N} \hat{u}_{0,i} \phi_{0,i} + \sum_{k=0}^{m_r} \sum_{l=1}^{n_k} \tilde{u}_{k,l} \psi_{k,l},
\]

\[
\tilde{q} = \sum_{i=1}^{N} \hat{Q}_i w_i = \sum_{i=1}^{N} \hat{q}_{0,i} \phi_{0,i} + \sum_{k=0}^{m_r} \sum_{l=1}^{n_k} \tilde{q}_{k,l} \psi_{k,l},
\] (4)

where \( \phi_{0,i} = \phi(\xi - j) \) and \( \psi_{k,l} = 2^k/2^l \psi(2^k \xi - l) \). These bases are used for the wavelet expansion on \( n_b \) subboundaries, and consist of \( n_s \) scaling functions and \( n_k \) wavelets at a resolution \( k \) on a subboundary. \( w_i (i = 1, \ldots, N, N; \text{DOF}) \) consist of \( \phi_{0,i} \) and \( \psi_{k,l} \). \( \hat{u}_{0,i}, \hat{u}_{k,l}, \hat{q}_{0,i}, \hat{q}_{k,l} \) are the expansion coefficients associated with \( u \) and \( q \). \( m_r \) is the finest resolution.

Substituting eqn (4) into (3) and applying the Galerkin method, we obtain the following boundary element equations:

\[
Hu = Gq,
\] (5)

where \( G \) and \( H \) are the matrices with elements given by

\[
g_{ij} = \int_{\Gamma} \int_{\Gamma} w_i w_j u^* \, d\Gamma^2, \quad h_{ij} = \frac{1}{2} \int_{\Gamma} \int_{\Gamma} w_i w_j \, d\Gamma + \int_{\Gamma} \int_{\Gamma} w_i w_j q^* \, d\Gamma^2
\]

\[(i, j = 1, \ldots, N).\] (6)
As a result, unknown components of \( u \) and \( q \) in eqn (5) can be obtained by solving the linear algebraic equation as

\[
Az = By = b,
\]

where \( z \) is the unknown vector, and \( y \) is the known vector of which components are evaluated by fast wavelet transform with \( O(N) \) operations from known values of \( u \) and \( q \). \( A \) and \( B \) are the coefficient matrices associated with \( z \) and \( y \), respectively.

### 2.3 Truncation strategy

In this study, the coefficient matrix entries are truncated before and after the calculation of eqn (6), in order to reduce the computational work for matrix generation.

The truncation before the computation, namely \textit{a priori} truncation, is carried out using the following estimation for the coefficients given by eqn (6):

\[
|g_{ij}| \simeq \tilde{g}_{ij} = \frac{\ell^n_{i} + 1 \ell^n_{j} + 1}{(m + 1)^{2-\beta_1-\beta_2}} \left\{ \frac{\alpha^m_n}{(m + n + 1)^{n+1}} \right\}^{\beta_1 + \beta_2} \cdot \frac{2^{-\frac{n}{2} + 1}}{\bar{r}n(\beta_1 + \beta_2)},
\]

\[
|h_{ij}| \simeq \tilde{h}_{ij} = \frac{\ell^n_{i} + 1 \ell^n_{j} + 1}{(m + 1)^{2-\beta_1-\beta_2}} \left\{ \frac{\alpha^m_n}{(m + n + 1)^{n+1}} \right\}^{\beta_1 + \beta_2} \cdot \frac{2^{-\frac{n}{2} + 1}}{\bar{r}n(\beta_1 + \beta_2) + 1},
\]

where \( \bar{r} \) is the distance between \( w_i \) and \( w_j \), and \( \ell \) is the length of a support of \( \phi \). \( \beta \) is given by \( \beta = 0 \) (\( w = \phi_0 \)) or 1 (\( w = \psi_k \)). If the estimations \( \tilde{g}_{ij} \) and \( \tilde{h}_{ij} \) satisfy

\[
\tilde{g}_{ij} < \eta_g \cdot G_{\text{max}}, \quad \tilde{h}_{ij} < \eta_h \cdot H_{\text{max}},
\]

then, \( g_{ij} \) and \( h_{ij} \) are truncated \textit{without} calculation of eqn (6). In eqn (9), \( G_{\text{max}} \) and \( H_{\text{max}} \) are the values that represent the maximum values of \( |g_{ij}| \) and \( |h_{ij}| \). \( \eta_g \) and \( \eta_h \) are the threshold parameters associated with the matrices \( G \) and \( H \). In the below, we will discuss determination of these parameters.

The elements not satisfying eqn (9) are computed by eqn (6), and after that, if \( g_{ij} \) and \( h_{ij} \) satisfy

\[
|g_{ij}| < \eta_g \cdot G_{\text{max}}, \quad |h_{ij}| < \eta_h \cdot H_{\text{max}},
\]

then, these elements are truncated \textit{a posteriori}.
3 Estimation of discretization and truncation error

To determine the optimal threshold for matrix compression, it is essential to estimate errors induced by discretization and truncation. This is because the optimal threshold can be determined such that the truncation error becomes comparable to the discretization one. In this study, these errors are estimated in the sense of residual, thus we will employ the residual $R$ for estimation of the discretization error.

3.1 Residual $R$

The residual $R$, proposed by Abe [5] for error estimation in adaptive mesh refinement, is defined as the vector $R$ in the following equation:

$$\mathbf{Ae} = \mathbf{R}$$

(11)

where $e$ is the vector of the error that stems from approximation, and components of the residual vector $R$, i.e. $R_i$ ($i = 1, \ldots, N$), are provided by

$$R_i = -\frac{1}{2} \int_{\Gamma} w_i(u - \hat{u}) \, d\Gamma - \int_{\Gamma} w_i(u - \hat{u})q^* \, d\Gamma^2 + \int_{\Gamma} w_i(q - \hat{q})u^* \, d\Gamma^2$$

(12)

where $\hat{u}$ and $\hat{q}$ are the wavelet series of true solutions $u$ and $q$, respectively. In general, since $u - \hat{u}$ and $q - \hat{q}$ cannot be evaluated exactly, $R_i$'s are computed approximately.

3.2 Truncation error

When the coefficient matrices in eqn (7) are compressed by the abovementioned truncation, the boundary element equation (7) can be rewritten by

$$\hat{\mathbf{A}}(z + \Delta z) = \hat{\mathbf{B}}\hat{\mathbf{y}}, \quad \hat{\mathbf{A}} = \mathbf{A} + \Delta \mathbf{A}, \quad \hat{\mathbf{B}} = \mathbf{B} + \Delta \mathbf{B},$$

(13)

where $\Delta z$ is the vector associated with a truncation error, and the matrices $\Delta \mathbf{A}$ and $\Delta \mathbf{B}$ consist of truncated elements of $\mathbf{A}$ and $\mathbf{B}$, respectively.

By eqns (5) and (13), the contribution of the truncation to residual, $\Delta \mathbf{A} \Delta \mathbf{z}$, is approximated by

$$\mathbf{A} \Delta \mathbf{z} \simeq -[\Delta \mathbf{A}] \mathbf{z} + [\Delta \mathbf{B}] \hat{\mathbf{y}} = -[\Delta \mathbf{H}] \mathbf{u} + [\Delta \mathbf{G}] \mathbf{q},$$

(14)

where $\Delta \mathbf{G}$ and $\Delta \mathbf{H}$ are the matrices containing the truncated entries of $\mathbf{G}$ and $\mathbf{H}$ in eqn (5).

By virtue of triangle inequality, norm of $\mathbf{A} \Delta \mathbf{z}$ is bounded as

$$||\mathbf{A} \Delta \mathbf{z}|| \leq ||\Delta \mathbf{H}|| \cdot ||\mathbf{u}|| + ||\Delta \mathbf{G}|| \cdot ||\mathbf{q}||.$$  

(15)
Substituting the relation \( q = G^{-1} H u \) in eqn (15), we obtain the following inequality:

\[
\frac{||A \Delta z||}{||u||} \leq ||H|| \left( \frac{||\Delta H||}{||H||} + \text{cond}(G) \frac{||\Delta G||}{||G||} \right), \tag{16}
\]

In the below, we consider eqn (16) in the sense of maximum norm. Since employment of wavelets enhances the asymptotical order of entries of \( G \) and \( H \), \( ||\Delta G|| \) and \( ||\Delta H|| \) can be estimated as follows:

\[
||\Delta G|| \simeq \eta_g \cdot G_{\text{max}} \simeq \eta_g ||G||, \quad ||\Delta H|| \simeq \eta_h \cdot H_{\text{max}} \simeq \eta_h ||H||. \tag{17}
\]

Substituting eqn (17) into eqn (16), we consequently obtain the estimation of truncation error as

\[
\frac{||A \Delta z||}{||u||} \leq ||H|| (\eta_h + \text{cond}(G) \cdot \eta_g). \tag{18}
\]

4 Determination of optimal threshold

The optimal value of the threshold is determined, provided that truncation error is comparable to discretization error, i.e. \( ||e_u|| \sim ||\Delta z_u|| \) associated with potential. This requirement however, cannot be considered directly, because the errors are estimated by residual, in this strategy. Hence, we employ \( ||Ae|| \sim ||A \Delta z|| \) as the necessary condition of determination of the optimal threshold, and assume that the sufficient condition \( ||e_u|| \sim ||\Delta z_u|| \) is satisfied under \( ||Ae|| \sim ||A \Delta z|| \).

The optimal threshold is determined a priori based on eqn (18). By the above necessary condition, we can approximately estimate the tolerance of residual under the optimal threshold as follows:

\[
\frac{||A \Delta z||}{||u||} = \frac{||Ae||}{||u||} = \frac{||R||}{||u||} \sim \frac{||c||}{||u||}, \tag{19}
\]

Validity of the threshold determined from eqn (19) will be investigated through numerical example. In eqn (19), the vector \( c \) is an approximation of \( R \), defined by

\[
c_i = -\frac{1}{2} \int_{\Gamma} w_i(\hat{u} - \bar{u}) \, d\Gamma - \int_{\Gamma} w_i(\hat{\Delta} - \bar{\Delta})q^* \, d\Gamma^2 \quad (i = 1, \ldots, N), \tag{20}
\]

where \( \hat{u} \) is a higher-order interpolation of \( u \) approximated by the scaling functions \( \phi_{m+1,i} \) \((i = 1, \ldots, N + n_b \cdot n_{m+1})\).

The value of \( \frac{||c||}{||u||} \) associated with a prescribed DOF \( N \) is not provided in advance. Hence, in order to estimate \( \frac{||c||}{||u||} \) approximately, we assume asymptotical behaviour of \( \frac{||c||}{||u||} \) as

\[
\frac{||c||}{||u||} \simeq \alpha \cdot N^{-\beta}, \tag{21}
\]
where $\alpha$ and $\beta$ are constants should be evaluated using two approximate solutions having different DOFs.

As will be shown in the example in Section 5, $\text{cond}(G)$ in eqn (18) can be replaced by unity, and then $\eta_g$ and $\eta_h$ have the same value $\eta$. Hence, replacing the sign of inequality in eqn (18) by that of equality and substituting eqns (19) and (20) into eqn (18), we obtain the optimal threshold as

$$\eta_{opt} = \frac{\alpha N^{-\beta}}{2||H||},$$

(22)

where $||H||$ is the maximum norm of the matrix $H$ that is used for determination of $\alpha$ and $\beta$ in eqn (21).

Since $\tilde{g}_{ij} \gg h_{ij}$ holds for $\tilde{r} \gg 1$, by the assumption $\eta_{opt} = \eta_g = \eta_h$, a priori truncation is carried out only using estimation $\tilde{g}_{ij}$ in eqn (8). If a certain $\tilde{g}_{ij}$ satisfies with eqn (9), $g_{ij}$ and $h_{ij}$ both are truncated.

5 Numerical example

Numerical example with boundary conditions depicted in Figure 1 is considered to verify the feasibility. In eqn (4), we employed Haar wavelets and piecewise linear wavelets ($n = 2$). The constants $\alpha$ and $\beta$ were estimated using approximations with 40 and 80 DOFs (Haar), and 45 and 85 DOFs (piecewise linear). The boundaries were divided into 5 subboundaries, as shown in Figure 1.

The present strategy requires two assumptions: (1) $||A\Delta z|| \sim ||Ae|| \sim ||c||$ at $||e_u|| \sim ||\Delta z_u||$; and (2) $\text{cond}(G)$ can be replaced by unity and then $\eta_{opt} = \eta_g = \eta_h$. In order to verify the validity of these assumptions, the relation between $\eta$ and three kinds of ratios $||A\Delta z||/||c||$, $||\Delta z_u||/||e_u||$ and $||A\Delta z||/(\eta||H|| \cdot ||u||)$ are shown in Figure 2. In this experiments, compres-
Figure 2: Relation between threshold \( \eta \) and three kinds of ratios \( \| A \Delta z \| / ||c|| \), \( || A \Delta z_u \| / || e_u || \) and \( || A \Delta z \| (\eta || H || \cdot || u ||) \). (Piecewise linear wavelets, 2565 DOFs, \( \circ \) : \( || A \Delta z \| / ||c|| \), \( \bullet \) : \( || A \Delta z_u \| / || e_u || \), \( \Delta \) : \( || A \Delta z \| (\eta || H || \cdot || u ||) \), \( \times \) : \( || u_{exact} - u || \))

(a) Truncation only of matrix \( G \).  
(b) Truncation only of matrix \( H \).

Figure 4 depicts convergence of \( ||c|| / ||u|| \).  
\( ||c|| / ||u|| \) has an asymptotical order \( O(N^{-\beta}) \) as assumed by eqn (21). 
Moreover, the parameters \( \alpha \) and \( \beta \) associated with its behaviour can be estimated accurately by extrapolation using numerical solutions having \( N = 30 \sim 100 \) DOFs. Since, in the application of wavelet BEM, \( N = 100 \) is very small relative to DOFs in usual problems, the computational cost for this estimation may be inexpensive.

Finally, compression rate of the coefficient matrix and CPU time under the optimal truncation are tabulated in Table 1. In this table, \( t_{est} \) and \( t_m \)
Figure 3: Asymptotical behaviour of $\|\tilde{H}\|/\|H\|$. $\|\tilde{H}\|$ is evaluated using the approximations with 64 (Haar) and 68 (piecewise linear) DOFs.

Table 1: Optimal threshold parameter $\eta_{opt}$ and computational performance.

<table>
<thead>
<tr>
<th></th>
<th>Haar</th>
<th>Piecewise linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>DOFs</td>
<td>2,560</td>
<td>2,565</td>
</tr>
<tr>
<td>$\eta_{opt}$</td>
<td>$1.74 \times 10^{-5}$</td>
<td>$1.70 \times 10^{-6}$</td>
</tr>
<tr>
<td>Rate of</td>
<td>1.94</td>
<td>1.45</td>
</tr>
<tr>
<td>compression (%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>$9.67 \times 10^{-4}$ ((\eta = \eta_{opt}))</td>
<td>$4.55 \times 10^{-4}$ ((\eta = \eta_{opt}))</td>
</tr>
<tr>
<td></td>
<td>$9.70 \times 10^{-4}$ ((\eta = 0))</td>
<td>$4.51 \times 10^{-4}$ ((\eta = 0))</td>
</tr>
<tr>
<td>$t_{est}$ (sec)</td>
<td>0.54</td>
<td>1.56</td>
</tr>
<tr>
<td>$t_m$ (sec)</td>
<td>7.39</td>
<td>15.44</td>
</tr>
</tbody>
</table>

denote the CPU time for determination of $\eta_{opt}$ and matrix generation, respectively. For approximations with about 2,500 DOFs, we obtained about 1% compression rate without loss of accuracy. The CPU time for determination of $\eta_{opt}$ was a few seconds, and it is very short compared with that for matrix generation. As a result, the present strategy enables us to determine an optimal threshold in wavelet BEM with a little computational work.

6 Conclusions

We have presented the determination of optimal threshold for wavelet BEM. The optimal threshold can be determined, provided that truncation error becomes comparable to a discretization one. In the present strategy,
these errors are estimated in the residual sense, for the sake of usefulness. Magnitude of the discretization error cannot be estimated \textit{a priori} without solving the boundary element equations, which is settled by use of approximation of the residual $R$ employed for estimation of the discretization error. Its approximation is given based on the asymptotical behaviour of potential contribution $c$ in the residual $R$. Since the asymptotical order of $||c||/||u||$ can be estimated using numerical solutions with small DOFs, computational cost for implementation of this strategy is inexpensive.

References


