



Indirect linear equation solvers for magnetostatics boundary element formulations

Z. Fang¹, M. S. Ingber² and M. J. Martinez³

¹*Department of Chemical Engineering, California Institute of Technology, USA*

²*Department of Mechanical Engineering, University of New Mexico, USA*

³*Multiphase Transport Processes Department, Sandia National Laboratories, Albuquerque, New Mexico, USA*

Abstract

A boundary element method (BEM) for the solution of magnetostatics problems is developed. For large problems when using a direct linear equation solver such as LU-decomposition, the decomposition phase of the algorithm comprises the predominant portion of the overall CPU time. A variety of indirect linear equation solvers are considered as a means of significantly reducing the CPU usage. Because of the excellent conditioning of the discretized linear systems, the number of iterations required of the indirect solvers to reach a prescribed residual is quite low and remains relatively constant with refined discretizations.

1 Introduction

The boundary element method (BEM) has become a well established tool for the analysis of three-dimensional electrostatic, magnetostatic, and eddy current problems. Integral representations for the magnetic flux density and the associated vector potential can be found in a variety of classical references [14, 10, 16, 1]. The BEM is particularly useful for exterior problems since only the surface of the domain needs to be discretized and radiation conditions are automatically satisfied. Classical boundary element formula-

tions for the magnetic vector potential have been developed by Rucker and Richter [12], Tsuboi and Misaki [15], and Adriaens *et al.* [1].

The current application of the BEM to the magnetic vector potential problem was developed to study material induction hardening problems [13]. In this implementation, the BEM in the exterior domain was coupled to a finite element method (FEM) to solve Maxwell's equations in solid parts which included nonlinear electromagnetic materials and heat transfer properties. There are several appealing aspects of using a combined FEM for the solid parts and the BEM for the exterior domain. For the so-called "scanning" problems, in which the part to be hardened moves through the induction coil, the coupled BEM/FEM approach simplifies the modeling by eliminating the need for a moving mesh. Even for stationary problems, the BEM is useful for handling the far-field radiation conditions. However, for large, 3D industrial problems, the solution of the dense BEM equations can be an impediment to the effective use of the coupled BEM/FEM approach. This provides the motivation for the current research to study a variety of iterative linear equation solvers for use with the vector potential BEM.

Several researchers have investigated the application of indirect linear equation solvers to discretized boundary integral equations. Kleinman [7] considered Dirchlet and Neumann problems for three-dimensional potential problems. Romate [11] investigated conjugate gradient type solvers applied to two-dimensional potential problems which resulted in Fredholm integral equations of the first or second kind. Kane *et al.* [6] considered generalized minimum residual (GMRES) Krylov-space iterative solvers applied to two-dimensional BEM elasticity analysis in single and multi-zone problems. They showed the importance of diagonal dominances of coefficient matrices in the performance of the iterative solvers. Mansur *et al.* [9] showed improved performance of a preconditioned biconjugate gradient and Lanczos solvers which outperformed Gaussian elimination for three-dimensional elasticity. Mammoli and Ingber [8] developed a physics-based preconditioner along with a GMRES solver for the multipole accelerated BEM analysis of suspension flows.

In the current work, a variety of indirect linear equation solvers are applied to the vector potential BEM formulation for magnetostatics problems with and without Jacobi preconditioning. Comparisons are made between the various indirect linear equation solvers and a traditional direct solver based on LU decomposition.

2 BEM formulation for magnetostatics

Following Stratton (1941), the equations governing magnetostatics problems are obtained by setting the time derivatives in Maxwell's equations to zero,

$$\nabla \times \mathbf{H} = \mathbf{J}, \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2)$$

where \mathbf{H} is the magnetic field intensity, \mathbf{J} is the current density, and \mathbf{B} is the magnetic induction, or magnetic flux density. In this case, conservation of charge is expressed by the steady continuity equation,

$$\nabla \cdot \mathbf{J} = 0. \quad (3)$$

Because the magnetic flux density is solenoidal, it can be expressed in terms of a vector potential as

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (4)$$

For a homogeneous and isotropic nonferromagnetic medium, the constitutive property is given by

$$\mathbf{B} = \mu \mathbf{H}, \quad (5)$$

where μ is the magnetic permeability. Combining Eqs. 1, 4, and 5 leads to the governing equation for the vector potential given by

$$\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J}. \quad (6)$$

The vector potential is uniquely defined if a "gauge" condition is specified. The gauge condition specified here is that the vector potential be solenoidal, that is,

$$\nabla \cdot \mathbf{A} = 0. \quad (7)$$

In the absence of a current density, (i.e., assuming $\mathbf{J} = \mathbf{0}$), the gauge condition then implies that

$$\nabla^2 \mathbf{A} = 0. \quad (8)$$

The governing vector Laplace equation can be transformed into a boundary integral equation using standard techniques, see, e.g., Stratton [14]. The vector potential can be represented at an arbitrary field point \mathbf{x} as

$$\begin{aligned} \eta(\mathbf{x})\mathbf{A}(\mathbf{x}) + \int_{\Gamma} \left[\frac{(\hat{\mathbf{r}} \cdot \mathbf{n})}{r^2} \mathbf{A} - \frac{(\hat{\mathbf{r}} \cdot \mathbf{A})}{r^2} \mathbf{n} + \frac{(\mathbf{A} \cdot \mathbf{n})}{r^2} \hat{\mathbf{r}} \right] d\Gamma(\mathbf{y}) = \\ \mu \int_{\Omega} \frac{\mathbf{J}}{r} d\Omega(\mathbf{y}) + \int_{\Gamma} \frac{\mathbf{B} \times \mathbf{n}}{r} d\Gamma(\mathbf{y}), \end{aligned} \quad (9)$$

where Ω is the domain of interest, $\hat{\mathbf{r}} = (\mathbf{x} - \mathbf{y})/r$ is the unit vector pointing from an arbitrary source point \mathbf{y} to the field point \mathbf{x} , $r = |\mathbf{x} - \mathbf{y}|$, and \mathbf{n} is the outward pointing unit normal vector to the boundary Γ at the source point \mathbf{y} . The coefficient $\eta(\mathbf{x})$ can be found by standard techniques such as off-diagonal summation. In the absence of volumetric current fluxes, the boundary integral forms a relation among boundary values of \mathbf{B} and \mathbf{A} . Hence, given the boundary magnetic flux, the boundary integral equation can be solved for the corresponding boundary values of the vector potential.

The BIE (Eq. 9) is discretized by constructing a boundary element mesh consisting of 9-node biquadratic isoparametric quadrilateral elements and 6-node quadratic isoparametric triangular elements. Quadratic isoparametric

elements allow the geometric approximation of general curved surfaces for 3D problems. Gaussian quadrature is used for evaluating the boundary integrals within the elements. Evaluation of the coefficient $\eta(\mathbf{x})$ is performed by row-summing the off-diagonal matrix coefficients.

3 Benchmark problem

A useful test problem for which an analytical solution is known is the uniformly magnetized sphere. Jackson [5] gives the solution expressed in spherical polar coordinates where M_o is the magnitude of the magnetization. The magnetic induction can be determined in terms of the vector potential using Eq. 4. The solution exterior to the sphere is given by

$$\begin{aligned} \mathbf{A} &= \frac{4\pi}{3} M_o \frac{a^3}{r^2} \sin \theta \mathbf{e}_\phi = \frac{4\pi}{3} M_o \frac{a^3}{r^2} (-y\mathbf{i} + x\mathbf{j}) \\ \mathbf{B} &= \frac{4\pi}{3} M_o \frac{a^3}{r^3} (2 \cos \theta \mathbf{e}_r + \sin \theta \mathbf{e}_\theta) \\ &= \frac{4\pi}{3} M_o \left(\frac{xz}{r^2} \mathbf{i} + \frac{yz}{r^2} \mathbf{j} + \frac{z^2}{r^2} \mathbf{k} \right) - \frac{4\pi}{3} M_o \frac{a^3}{r^3} \mathbf{k}. \end{aligned} \quad (10)$$

An error analysis for this problem on a variety of grids is presented in Table 1. The L_2 error norm is defined by

$$\|E\|_2 = \left(\frac{1}{3N} \sum_{i=1}^3 \sum_{I=1}^N (A_{i,I} - A_{i,I}^h)^2 \right)^{1/2}, \quad (11)$$

where $A_{i,I}^h$ is the numerical solution component in the i -th direction at the I -th grid point. Since the norm of the x and y components of the vector potential are of order unity (about 4), the results indicate just below 1% relative error for $N = 978$, and less than 0.5% error for $N = 2034$, where N denotes the matrix order (3 times the number of grid points).

As expected, the results show the matrix build time is of order N^2 while the LU decomposition time is of order N^3 . The first two mesh refinements show only a superlinear convergence rate with an exponent of roughly 1.4. The solutions for the finest discretization show the convergence to have slowed, most likely due to finite arithmetic errors in solving the linear system. These calculations were performed using double-precision arithmetic on a SUN ES6000 (250MHz) system.

4 Iterative solution of dense BIEM matrices

The CPU time for the assembly portion of the BEM algorithm was shown in the previous section to scale as N^2 whereas the solution of the linear equations using LU decomposition scales as N^3 . Clearly, for large systems,

Table 1: Error Analysis Using a Direct Solver

Matrix order, N	Matrix build time	LU time	$\ E\ _2$
306	3.4	0.777	0.1428
978	34.4	28.2	0.02758
2034	156.6	267.6	.01050
3678	490.2	1642.9	.01102

the solution phase of the algorithm will dominate for direct solvers. Iterative solvers have the potential to reduce the scaling of the solution phase to N^2 , assuming the number of iterations does not increase significantly for finer discretizations. In the following, the efficiency of using iterative methods for solving the linear equations associated with the vector potential is investigated.

Most packages for iterative solution methods, such as Aztec [4] or PETSc [2], are designed for sparse systems, employing some form of sparse matrix storage format. Application of a sparse matrix storage technique to a dense system will severely compromise the efficiency of the package because of the substantial overhead involved in indirect addressing for basic linear algebra operations, such as matrix-vector multiplies. In the present work, an iterative linear equation solving package due to Barrett *et al.* [3] is used which does not use any condensed storage format. The package includes many of the popular acceleration methods for non-symmetric systems such as the bi-conjugate gradient (BiCG), conjugate gradient squared (CGS), bi-conjugate gradient stabilized (BiCGstab), and quasi-minimal residual (QMR) methods along with a Jacobi preconditioner. All these iterative solvers require the user to provide their own matrix-vector multiplication subroutines.

The choice of the stopping criteria for iterative solution methods is an important consideration. The primary and most common stopping criteria is based on residual reduction. The residual at the k th iteration step, r^k , is defined by

$$\|r^k\| = \|b - Ax^k\| \quad (12)$$

where A is the coefficient matrix, b is the right-hand-side vector, x^k is the k th trial solution, and L_2 norms are applied. A convenient stopping criterion is given by

$$\|r^k\| < \epsilon \|b\| \quad (13)$$

for some suitably small ϵ .

The verification problem (uniformly magnetized sphere) is not used in this subsection to test the convergence of the iterative solvers. The reason for this is that, because of the excellent conditioning of the associated linear systems, the number of iterations until convergence was typically less than

5. The problem considered here is comprised of two uniformly magnetized spheres separated by a distance d as shown in Figure 1. Again, Neumann boundary data is prescribed. By superposition, the exterior solution can be shown to be

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= \left(\frac{4\pi}{3} M_0 a^3 \frac{\sin \theta}{r^2} + \frac{4\pi}{3} M_0 a^3 \frac{r \sin \theta}{R^3} \right) \mathbf{e}_\phi \\ &= \frac{4\pi}{3} M_0 a^3 \left[- \left(\frac{1}{r^3} + \frac{1}{R^3} \right) y \mathbf{i} + \left(\frac{1}{r^3} + \frac{1}{R^3} \right) x \mathbf{j} \right] \end{aligned} \quad (14)$$

and

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \frac{4\pi}{3} M_0 a^3 \left[\frac{2 \cos \theta}{r^3} + \frac{2 \cos \theta (r^2 + d^2) - rd \cos^2 \theta - 3rd}{R^5} \right] \mathbf{e}_r \\ &+ \frac{4\pi}{3} M_0 a^3 \left[\frac{\sin \theta}{r^3} + \frac{\sin \theta (r^2 - 2d^2 + rd \cos \theta)}{R^5} \right] \mathbf{e}_\theta \\ &= 4\pi M_0 a^3 \left[\frac{xz}{r^5} + \frac{x(z-d)}{R^5} \right] \mathbf{i} \\ &+ 4\pi M_0 a^3 \left[\frac{yz}{r^5} + \frac{y(z-d)}{R^5} \right] \mathbf{j} \\ &+ 4\pi M_0 a^3 \left[\left(\frac{z^2}{r^5} - \frac{1}{3r^3} \right) + \left(\frac{(z-d)^2}{R^5} - \frac{1}{3R^3} \right) \right] \mathbf{k} \end{aligned} \quad (15)$$

where

$$R = (r^2 + d^2 - 2dz)^{\frac{1}{2}}.$$

A variety of residuals, ϵ , were chosen for terminating the iteration for the indirect solvers. It was determined that both the L2-error norm and the sup-error norm did not change for $\epsilon < E-06$ for any of the indirect solvers. Hence, $\epsilon = 1.0E-06$ was chosen for the residual in all subsequent calculations. Two different separations d (see Figure 1) were considered, namely, $d = 4.0$ and $d = 2.1$. A convergence study was first performed to test the accuracy of the two-sphere problem. The error norm is shown as a function of the degrees of freedom in Figure 2. All indirect solvers yielded essentially the same error norms, and hence, only one indirect solver norm is shown in the Figure 2. It is somewhat curious to note that for some unknown reason the indirect methods consistently yielded smaller L2-norm errors than the direct methods. The error is seen to be less for the greater separation, $d = 4.0$, compared to $d = 2.1$ as expected.

The CPU times for the linear equation solution portion of the algorithm along with iteration counts for the various solvers are shown in Tables 2 and 3 for the two sphere problem with separations of $d = 4.0$ and $d = 2.1$, respectively. The number of degrees of freedom considered for both tables is $N = 4380$. In all cases, the CPU time for the matrix assembly was approximately 49 CPU seconds. The excellent convergence rates is evident

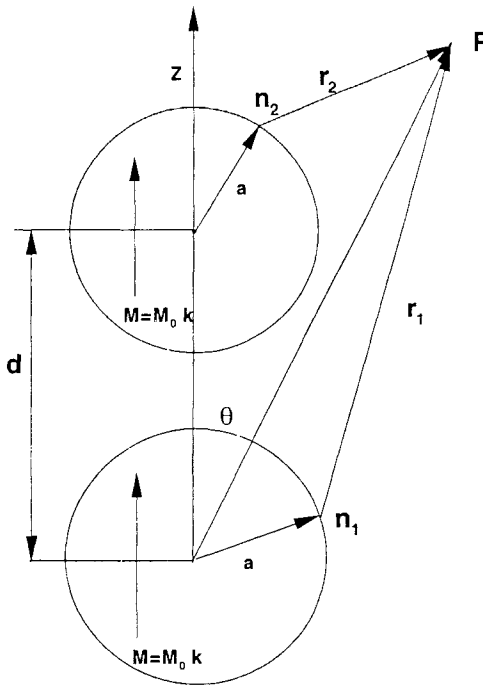


Figure 1: Problem geometry for the two uniformly magnetized spheres problem.

for indirect solvers with the largest number of iterations being 9 for the CGS solver. In fact typically, Jacobi preconditioning did not improve the convergence of the indirect solvers, and in some cases, actually caused a slight increase in the number of iterations. The reason for this is that the linear systems were very well conditioned. As an example for the finest discretization considered with $N = 4380$, the condition number for the case $d = 2.1$ was 5630 and the condition number for the case $d = 4.0$ was 2133. The iteration count is seen to increase for the case of $d = 2.1$ compared to the case of $d = 4.0$. This is a result of the increase of condition number. The number of iterations as a function of the degrees of freedom is shown in Figure 3. It is interesting to note that the number of iterations for a given indirect solver remained essentially constant as the problem size increased. This indicates that the solution phase of the algorithm truly does scale as the number of degrees of freedom squared for the indirect solvers.

Finally, the CPU times are seen to be substantially reduced for the indirect solvers compared to the direct solver. The iterative solvers are

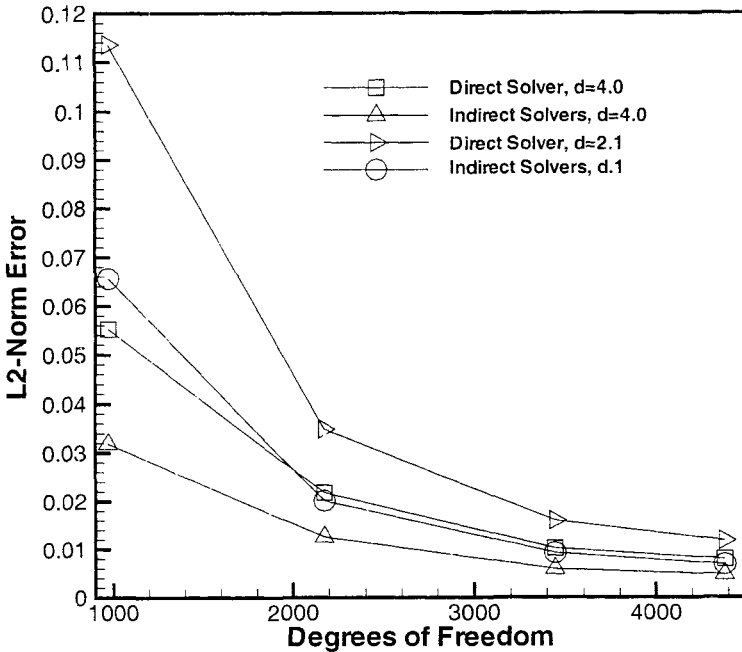


Figure 2: The L2-norm error as a function of degrees of freedom, N , for the two-sphere problem.

anywhere from 25 (for the smallest problem considered here) to 140 (for the largest problem) times faster than the LU solver.

5 Conclusions

A BIEM for magnetostatics problems in 3D geometries has been developed and benchmarked. Solution of the linear system via LU decomposition has been applied for systems as large as 4380 equations. These require about 19.1 Mwords of memory, 1.32 CPU-minutes to assemble and 12.4 CPU-minutes to solve the matrix equations on a DEC-Alpha 500 MHz machine. Iterative methods were investigated in order to reduce the CPU time for the solution of large problems. The iterative methods appear to provide an efficient alternative to LU decomposition. All problems considered were Neumann problems which resulted in Fredholm integral equations of the second kind and well-conditioned discretized linear equations.

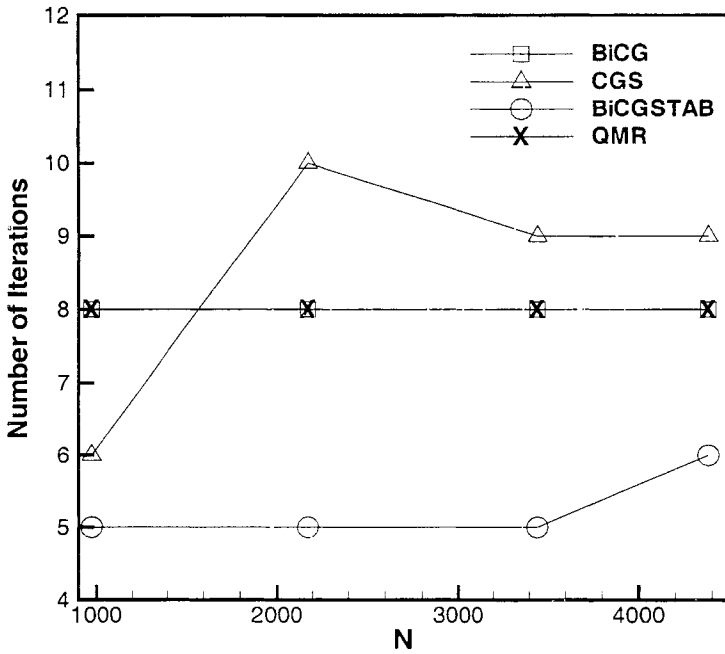


Figure 3: The number of iterations required for convergence of the iterative solvers as a function of degrees of freedom, N , for the two-sphere problem.

Table 2: Performance of iterative solvers for $N = 4380$, $d = 4.0$.

Method	Residual	Jacobi Preconditioning	Iterations	Solution CPU
LU				717
BiCG	9.77E-08	no	6	5.19
BiCG	2.07E-07	yes	6	5.18
CGS	2.00E-07	no	4	4.24
CGS	1.19E-07	yes	4	4.24
BiCGSTAB	5.61E-08	no	4	4.24
BiCGSTAB	7.08E-08	yes	4	4.24
QMR	9.83E-08	no	6	5.19
QMR	2.00E-07	yes	6	5.19

Table 3: Performance of iterative solvers for $N = 4380$, $d = 2.1$.

Method	Residual	Jacobi		Solution CPU
		Preconditioning	Iterations	
LU				742
BiCG	5.32E-07	no	8	6.92
BiCG	6.31E-07	yes	8	6.92
CGS	5.38E-08	no	9	9.56
CGS	2.39E-07	yes	8	8.50
BiCGSTAB	6.11E-08	no	6	6.38
BiCGSTAB	2.42E-07	yes	5	5.31
QMR	4.61E-07	no	8	6.92
QMR	7.37E-07	yes	8	6.91

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