Structural dynamics using Gaussian mass matrix

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Abstract
In this paper, a new technique for dynamic structural analysis is presented. The present technique is based on using the dual reciprocity boundary element method. The inertia term is approximated using the Gaussian radial bases functions. Only the problem boundary is needed to be discretized. Internal degrees of freedom could be added to improve the solution accuracy. The particular solutions which are necessary to form the mass matrix are derived and given in explicit forms. The singular case is studied and the limiting values of these solutions are derived. An example problem is presented to validate the accuracy of the present formulations.

1 Introduction
By the improvements of the structural analysis techniques, it is common nowadays to require monitoring the behavior of a structure under dynamic loading. In the former 30 years, the superiority of the Boundary Element Method (BEM) [1] is understood among researchers in numerical modelling as it requires only the discretization of the problem boundary and its ability to model infinite domains. The work of Cruse and Rizzo [2] can be considered as one of the early publications in the application of the BEM in dynamics. Dynamic analysis using the BEM can be categorized into three categories, the time domain formulation, the integral transform technique and the mass matrix technique. In time domain formulation, a dynamic-type

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fundamental solution has to be dealt with, leading to more complicated mathematics. Moreover the accuracy of the results are dependent on the choice of the time step. Similar complications happened when dealing with the integral transform technique. Moreover in this case some numerical inaccuracies happen in performing the transformation inverse. Both time and integral transform techniques are computational expensive as going forward in the time history. The third technique is based on using boundary-interior element approach which leads to an easy formulation and finite element-like equations are obtained. However such formulation requires the discretization of both the boundary and the domain of the considered problem. An extensive survey for such techniques is given by Dominguez [3] and Beskos [4].

In the early 1980s, Nardini and Brebbia [5] developed an easy technique to treat elasto-dynamics problems. This technique employs the static type fundamental solution as that which is used in the boundary-interior element technique. Inertia terms are treated as pseudo body forces which can be dealt with using particular solutions. In such case the inertia term is dependant on the problem acceleration which is not prescribed. A finite difference technique is used to interpret the acceleration in terms of the unknown displacements [6]. An approximation procedure based on suitable functions together with collocation scheme is used to compute the unknown particular solutions. The technique named later as the dual reciprocity method or the mass matrix approach. The main advantage of this technique is the simplicity as only the boundary of the considered problem is needed to be discretized. The mass matrix technique does not require the same computational effort as that required by the formerly mentioned techniques.

Since the publication of the first paper by Nardini and Brebbia [5], many researchers has extended the technique to solve many problems. A survey for such applications is given by Partridge et al. [7]. It has to be noted that, in most of the above publications, the conventional function \((c+R; c: \text{ is constant})\) was used as the approximation function which is known as conical radial basis functions.

In this paper, the Gaussian radial basis functions are used as the approximation functions for transient dynamic analysis of structures. The necessary particular solutions corresponding to these functions are derived, and implemented into computer program. The limiting
case at the singular collocation point is studied and the limiting values of the derived particular solutions are computed and given in explicit form. Two numerical examples are presented to test the accuracy and the validity of the present formulation.

2 Transient dynamic analysis

Consider the elastic body shown in figure (1). The body has a domain $\Omega$ and boundary $\Gamma$. The displacements are denoted by $u_j$ where $j = 1, 2$. The equilibrium equations in terms of displacements are given by (in the absence of body forces) [3]:

$$L_{ij}u_j - \rho \ddot{u}_i = 0$$

(1)

in which the tensor notation is used: comma denotes derivative with respect to spatial coordinates, $L_{ij}$ is the Navier differential operator [1], $\rho$ is the body density and the over dots denote derivatives with respect to time; therefore $\ddot{u}$ denotes the body acceleration. The solution of equation (1) can be split into two parts the complementary solution $u_i^c$ and the particular integral part $u_i^p$ where:

$$L_{ij}u_j^c = 0 \quad \text{and} \quad L_{ij}u_j^p = \rho \ddot{u}_i$$

(2)
The integral representation of the complementary solution can be written as follows [1]:

\[ C_{ij}u_j^c + \int_\Gamma T_{ij}u_j^c d\Gamma = \int_\Gamma U_{ij}t_j^c d\Gamma \quad (3) \]

where \( C_{ij} \) is the jump term, \( C_{ij} = \delta_{ij}/2 \) if the collocation point is located on a smooth boundary point and \( C_{ij} = \delta_{ij} \) if the collocation point is an internal point. The integral sign with the dash denotes Cauchy principal value integral. The kernels \( T_{ij} \) and \( U_{ij} \) are the two-point Kelvin fundamental solution [1].

Substituting in equation (3) by \( u_j^c = u_i - u_i^p \) and \( t_j^c = t_i - t_i^p \), it gives:

\[ C_{ij}u_j + \int_\Gamma T_{ij}u_j^c d\Gamma = \int_\Gamma U_{ij}t_j^c d\Gamma + \left[ C_{ij}u_i^p + \int_\Gamma T_{ij}u_i^p d\Gamma - \int_\Gamma U_{ij}t_i^p d\Gamma \right] \quad (4) \]

The expressions of the particular solutions are given according the following collocation scheme [5]:

\[ u_j^p = \sum_{k=1}^m \Psi_{jl}^k \alpha_j^k \quad (5) \]

where \( \alpha_j^k \) are unknown coefficients at another set of field points \( y_k \in \Gamma \) (see figure (1)) and \( \Psi_{jl}^k \) is a suitable displacement kernel, which will be defined later. The integral equation in equation (4) can be re-written as follows:

\[ C_{ij}u_j + \int_\Gamma T_{ij}u_j^c d\Gamma = \int_\Gamma U_{ij}t_j^c d\Gamma + \sum_{k=1}^m \left[ C_{ij}\Psi_{jl}^k + \int_\Gamma T_{ij}\Psi_{jl}^k d\Gamma - \int_\Gamma U_{ij}\eta_{jl}^k d\Gamma \right] \alpha_j^k \quad (6) \]

where \( \eta_{jl}^k \) is the corresponding traction kernel to the displacement field \( \Psi_{jl}^k \) at any point \( y_k \) and is given by:

\[ \eta_{jl}^k = \mu \left[ \Psi_{jl,\beta} + \Psi_{jl,\beta,i} + \frac{2\nu}{1-2\nu}\delta_{i\beta}\Psi_{jl,\theta,\theta} \right] \eta_{\beta}^k \quad (7) \]

where \( \nu \) is the Poisson’s ratio. It is important to note that a transposition is introduced to the indices in the former equation to allow the multiplication between the kernels in equation (6).
3 Particular solutions

Assuming that the inertia term is approximated using the following functions (recall equation (5)):

\[ \rho \ddot{u}_i = \sum_{k=1}^{m} f^k c_i^k \]  

(8)

From equations (2) and (5) the following equation can be obtained:

\[ L_{li} \Psi_{ij}^k = f^k \delta_{lj} \]  

(9)

where \( f^k \) can be chosen as any function. If \( f^k \) is chosen according to the following Gaussian function \( f^k = -e^{-R_k^2} \), where \( R_k \) denotes the distance between the new source point \( x \) and the new fields point \( y_k \) (see figure (1)). The comma denotes derivative with respect to the coordinate of the field point \( y_k \). The expression of the kernel \( \Psi_{ij}^k \) can be evaluated using operator de-coupling via Hörmander method which is equivalent to the use of the Galerkin vector as follows [1]:

\[ \Psi_{ij}^k = G_{ij,\theta \theta}^k - \frac{1}{2(1 - \nu)} G_{ij,\theta \theta}^k \]  

(10)

and the corresponding Galerkin vector can be computed from the following equation:

\[ C_{ij}^k = \frac{1}{\mu} g_k \delta_{ij} \quad \text{and} \quad \nabla^4 g_k = f^k = -e^{-R_k^2} \]  

(11)

where \( \mu \) is the modulus of shear, \( g_k \) is a particular solution. The \( \nabla^4 = \nabla^2 \nabla^2 \) denotes the two-dimensional bi-harmonic operator and \( \nabla^2 \) denotes the two-dimensional Laplace operator. A suitable particular solution for this equation is:

\[ g_k = \frac{1}{16} \left[ -e^{-R_k^2} - R_k^2 \left( 1 + R_k^2 \right) \left( E_1 (R_k^2) + \ln R_k^2 \right) \right] \]  

(12)

in which \( E_1(\cdot) \) is the exponential integral [8]. Using equations (10) and (11), the displacement particular solutions can be computed as follows:

\[ \Psi_{ij}^k = \frac{1}{\mu} \delta_{ij} \left[ \frac{g_k'}{R_k} + g_k'' \right] - \frac{1}{2(1 - \nu)\mu} \left[ \frac{g_k'}{R_k} \left( \delta_{ij} - R_{k,i} R_{k,j} \right) + g_k'' R_{k,i} R_{k,j} \right] \]  

(13)
where the relevant derivatives of \( g_k \) can be computed as follows:

\[
g'_k = \frac{1}{8} \left[ -e^{-R_k^2} + \frac{1}{R_k} + R_k E_1(R_k^2) + R_k \ln R_k^2 \right] \\
g''_k = \frac{1}{8} \left[ \frac{2R_k^2 - 1 + e^{-R_k^2}}{R_k^2} + E_1(R_k^2) + \ln R_k^2 \right]
\]  

Using equation (7)', the corresponding traction particular solution kernels can be computed as follows:

\[
\eta^k_{ij} = N_j R_{k,i} \left[ \frac{\nu}{1 - \nu} \left( \frac{g'_k}{R_k^2} - \frac{g''_k}{R_k} \right) + g'''_k \right] + \\
N_i R_{k,i} \left[ \left( \frac{g'_k}{R_k^2} - \frac{g''_k}{R_k} \right) + \frac{\nu}{1 - \nu} g'''_k \right] + \\
\delta_{ij} R_{k,N} \left[ \frac{\nu}{1 - \nu} \left( \frac{g'_k}{R_k^2} - \frac{g''_k}{R_k} \right) + g'''_k \right] - \\
R_{k,i} R_{k,j} R_{k,N} \frac{1}{1 - \nu} \left[ 3 \left( \frac{g'_k}{R_k^2} - \frac{g''_k}{R_k} \right) + g'''_k \right]
\]  

(15)

where \( N_i \) denotes the components of the normal at the field point \( y_k \), \( R_{k,N} = R_{k,i} N_i \) and:

\[
g'''_k = \frac{-(1 + 2R_k^2)e^{-R_k^2} + (1 + R_k^2)}{4R_k^3}
\]  

(16)

4 Limiting case \((R_k \to 0)\)

In order to successfully perform the collocation scheme to approximate the inertia term, a limiting study has to be performed first as \( R \) approaching zero. The following limits can be easily proven:

\[
\lim_{R_k \to 0} g'' = \frac{1 - \gamma}{8}, \quad \lim_{R_k \to 0} g'''_k = 0, \quad \lim_{R_k \to 0} \frac{g'_k}{R_k} = \frac{1 - \gamma}{8}, \quad \lim_{R_k \to 0} \left( \frac{g''_k}{R_k^2} - \frac{g'_k}{R_k^2} \right) = 0 \quad \text{and} \quad \lim_{R_k \to 0} \left( \frac{g'_k}{R_k} - g''_k \right) = 0
\]  

(17)

Where \( \gamma \) is the Euler-Gamma constant [8]. Substituting with the above limits into equations (13) and (15), it can be easily shown when \( R_k \) approaches zero, the expressions of the particular solution kernels are reduced to the following forms:

\[
\lim_{R_k \to 0} \Psi^k_{ij} = \frac{1 - \gamma}{16 \mu} \delta_{ij} \frac{3 - 4\nu}{1 - \nu} \quad \text{and} \quad \lim_{R_k \to 0} \eta^k_{ij} = 0
\]  

(18)
5 Numerical implementation

In order to numerically solve the integral equation in equation (6), the boundary is discretized into elements. Quadratic elements are assumed to be used in this work. The point collocation technique [1] is used in this process. For each collocation point $\xi = x_i$, equation (6) can be re-written as follows:

\[
Hu = Gt + [H\Psi - G\eta]\alpha
\]  

(19)

where $H$ and $G$ are the well-known boundary element influence matrices [1]. $u$ and $t$ are the vectors of the boundary displacements and tractions respectively. $\Psi$ and $\eta$ are the matrices of the particular solutions and $\alpha$ is the vector of the unknown collocation coefficients.

Consequently equation (8) can be re-written as follows:

\[
\rho \ddot{u} = F\alpha
\]  

(20)

where $F$ is a collocation matrix. In order to get square matrix, the number of the field points in the second collocation scheme $y_k$ will be chosen to be equal to number of nodes. By pre-multiplying both sides of the former equation by $F^{-1}$, the following equation is obtained:

\[
\alpha = \rho F^{-1}\ddot{u}
\]  

(21)

The main advantage of using the Gaussian function is the $F$ matrix is sparse and easily to get its inverse without any ill-conditioning cases which are common when using other global based function. Substitute equation (21) into equation (19), it gives:

\[
Hu = Gt + \rho[H\Psi - G\eta]F^{-1}\ddot{u}
\]  

(22)

or in compact form:

\[
Hu = Gt + M\ddot{u}
\]  

(23)

where $M$ is the mass matrix. It has to be noted that in the above equation both of the displacement and the traction vectors $u = u_t$ and $t = t_t$ are defined at a certain time $t$. In order to solve the above equation, it is necessary to express $\ddot{u}$ in terms of $u$. This can be done easily by using the following Houbolt finite difference scheme [6]:

\[
\ddot{u}_{t+\Delta t} = \frac{2u_{t+\Delta t} - 5u_t + 4u_{t-\Delta t} - u_{t-2\Delta t}}{\Delta t^2}
\]  

(24)
where $\Delta t$ is the time step. Substitute equation (24) into equation (23), it gives:

$$\left[ \frac{2}{\Delta t^2} M + H \right] u_{t+\Delta t} = G_{t+\Delta t} + \frac{1}{\Delta t^2} M \{ u_t - 4u_{t-\Delta t} + u_{t-2\Delta t} \}$$

(25)

As it can be seen that this equation can be solved for the unknown boundary displacements and tractions at time $t + \Delta t$.

6 Internal nodes

The former approximation for the inertia term in terms of collocation based particular solutions are based on distributing the field points $y_k$ on the boundary. Such distribution, however, sometimes cannot achieve the required accuracy; therefore it is more appropriate to add internal field points (or internal nodes) to improve the accuracy of the solution. By adding a certain number of internal nodes, equation (23) can be re-written as follows:

$$\begin{bmatrix} H_{BB} & 0 \\ H_{DB} & I \end{bmatrix} \begin{bmatrix} u_B \\ u_D \end{bmatrix} = \begin{bmatrix} G_{BB} \\ G_{DB} \end{bmatrix} \{ t_B \} + \begin{bmatrix} M_{BB} & M_{BD} \\ M_{DB} & M_{DD} \end{bmatrix} \begin{bmatrix} \ddot{u}_B \\ \ddot{u}_D \end{bmatrix}$$

(26)

where $0, I$ are the zero and the unity matrices respectively, the subscripts $B$ and $D$ denote boundary and internal nodes respectively, and the mass matrix is given by:

$$\begin{bmatrix} M_{BB} & M_{BD} \\ M_{DB} & M_{DD} \end{bmatrix} = \begin{bmatrix} H_{BB} & 0 \\ H_{DB} & I \end{bmatrix} \begin{bmatrix} \Psi_{BB} & \Psi_{BD} \\ \Psi_{DB} & \Psi_{DD} \end{bmatrix} -$$

$$\begin{bmatrix} G_{BB} \\ G_{DB} \end{bmatrix} \begin{bmatrix} \eta_{BB} & \eta_{BD} \end{bmatrix} \begin{bmatrix} F_{BB} & F_{BD} \\ F_{DB} & F_{DD} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_B \\ \alpha_D \end{bmatrix}$$

(27)

7 Example

In order to demonstrate the accuracy of the present formulation an example is presented. The results are compared to previously published results based the conventional $(c + R)$ function and finite element methods. The numerical integrations are performed using the Gauss-Legendre scheme with 4 points.
7.1 Frame-like structure

In this example the frame-like structure shown in figure (2) is considered. This example was considered by Nardini and Brebbia [5] with the following properties: \( E/\rho = 10^4 \text{m} \) and \( \nu = 0.2 \). The structure is loaded by a wind-like loading as shown in figure (2) according to the shown loading function. The structure is discretized into 12 boundary elements. Two sets of internal points are considered: 3 and 5 internal points (see figure (2)). The results for the horizontal displacement at point "A" and the horizontal and vertical tractions at nodes "B" and "C" are plotted in figures (3) to (5). The results are plotted together with the results obtained from Refs. [5], [3] using the \((c + R; c = 1)\) function. Another set of results for the horizontal displacement at point A is plotted in figure (3) using a finite element analysis (see Ref. [5]) using the Wilson-\( \theta \) approach (see Ref. [6]). \( \Delta t \) is taken
Figure 3: Horizontal displacement at point "A" in the frame-like structure.

to be equal to 0.005 to allow comparison to previously published results. The agreements between the results are shown in these figures. It has to be noted that the \((c + R)\) function is more accurate in this case as the results of the Gaussian function converge to the results of that function by increasing the number of the internal nodes. This is mainly because the Gaussian function is a local function; therefore it will not gives the best approximation for the inertia term as that of a global based function such as the \((c + R)\) function especially for complex geometry. However, the resulting collocation matrix is sparse in the case of using the Gaussian function. This can be regarded as the best advantage when using the Gaussian function.

8 Conclusions

In this paper, the transient elasto-dynamic problems are solved using static-type boundary element solution. The inertia terms are treated as pseudo body forces via particular solutions. The Gaussian radial bases functions are used to approximate such particular solutions. A collocation procedures were used to compute the approximation coefficients. The Mass matrix is formed and a step-by-step solution is presented for the time approximation. The main advantage of using the Gaussian function is the resulting collocation matrices are sparse
Figure 4: Horizontal traction at nodes "B" and "C" in the frame-like structure.

Figure 5: Vertical traction at nodes "B" and "C" in the frame-like structure.
and well-conditioned, which do not require huge computer storage. The results of the analyzed example confirm the accuracy and the validity of the present formulation.

References


