Transient dynamic response of 3-D elastoplastic structures by the D/BEM

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Abstract

A general boundary element methodology is presented for the transient dynamic analysis of three-dimensional elastoplastic solids and structures. The elastostatic fundamental solution is employed in the integral formulation of the problem and this creates inertial volume integrals in addition to the nonlinear volume integrals due to inelasticity. Thus an interior discretization in addition to the usual surface discretization is necessary. Isoparametric linear quadrilateral elements are used for the surface discretization and isoparametric linear hexahedra for the interior discretization. Advanced numerical integration techniques for singular and nearly singular integrals are employed. Houbolt's step-by-step numerical time integration algorithm is used to provide the dynamic response. Numerical examples are presented to illustrate the method and demonstrate its accuracy.

1 Introduction

For many types of structures the use of linear structural analysis is no longer considered adequate for achieving a realistic and safe design. To obtain more insight into the behavior, one must take into account nonlinearities due to inelasticity of the material. In recent decades, significant research has been carried out in an effort to develop reliable numerical methods for determining the nonlinear behavior under transient dynamic loading of various types of structures. Some of the important applications of inelastic dynamic analysis are found in the design of nuclear reactors, ship structures, transportation vehicles and multistory buildings subjected to earthquake loadings. The finite element
method (FEM) is the most popular numerical method for the solution of inelastic dynamic problems involving two- and three-dimensional (2-D and 3-D) solids and structures [1]. Recently, the boundary element method (BEM) has emerged as a reliable alternative method of solution of this class of problems, as it is evident in the review article of Beskos [2].

The BEM in its direct conventional form and in conjunction with the elastostatic fundamental solution of the problem has been successfully used for the analysis of 2-D elastoplastic solids and structures under dynamic loading [3-5]. The BEM in its symmetric Galerkin form and in conjunction with the elastostatic fundamental solution of the problem has also been successfully applied to 2-D dynamic elastoplastic problems [6]. The BEM in its direct conventional form and in conjunction with the elastodynamic fundamental solution of the problem appears to be complicated and time-consuming [7,8]. Thus, the need for the development of an accurate and efficient BEM for the elastoplastic analysis of 3-D structures under dynamic loading is apparent.

In this paper a general boundary element methodology is developed for the analysis of three-dimensional elastoplastic solids and structures under dynamic loads. The method employs the elastostatic fundamental solution of the problem because of its simplicity. This creates volume integrals due to inertia and inelasticity in addition to the boundary ones. Thus, an interior discretization is necessary in addition to the boundary one. The matrix equations of motion are numerically integrated in time with the aid of Houbolt’s algorithm. Two numerical examples serve to illustrate the method and demonstrate its accuracy.

2 Boundary element formulation and solution

For a three-dimensional body \( \Omega \) which is bounded by its surface \( \Gamma \), the Somigliana identity for the dynamic elastoplastic case associated to the initial stress formulation is defined as

\[
c_{ij}u_{ij}(\xi, t) = \int_{\Gamma} u_{ij}^*(\xi, X)p_{ij}(X, t)d\Gamma(X) - \int_{\Gamma} p_{ij}^*(\xi, X)u_{ij}(X, t)d\Gamma(X) - \rho \int_{\Omega} u_{ij}^*(\xi, X)\dot{u}_{ij}(X, t)d\Omega(X) + \int_{\Omega} \varepsilon_{ij}^*(\xi, X)\sigma_{ij}^p(X, t)d\Omega(X)
\]

In the above, \( t \) is the time, \( \rho \) the constant mass density of the body, \( c_{ij} \) the usual free coefficient of elastostatic analysis and \( u_{ij}^*(\xi, X), p_{ij}^*(\xi, X) \) and \( \varepsilon_{ij}^*(\xi, X) \) are the fundamental solution components of the elastostatic problem representing the displacement, traction and strain, respectively. Furthermore, \( u_{ij}, \dot{u}_{ij}, p_{ij} \) and \( \sigma_{ij}^p \) represent the displacements, accelerations, tractions and inelastic stresses, respectively. Eqn (1) represents the equation of motion of the body in integral form. For \( \rho = 0 \) this equation is reduced to the static case. In order to solve this equation, the boundary element method (BEM) is applied. The boundary of the 3-D body is discretized into \( NB \) linear quadrilateral boundary elements and the domain is discretized into \( NV \) linear hexahedral volume cells. Then, eqn (1) becomes
\[ c_{ij} u_j(\xi, t) = \sum_{m=1}^{N_B} \left\{ \int_{\Gamma_m} u_i^*(\xi, X) \Phi \, d\Gamma \right\} p_j(X, t) - \sum_{m=1}^{N_B} \left\{ \int_{\Gamma_m} p_i^*(\xi, X) \Phi \, d\Gamma \right\} u_j(X, t) \]

Eqn (2) is rewritten in a compact form as

\[ c_{ij} u_j = \sum_{m=1}^{N_B} G_{ij} p_j - \sum_{m=1}^{N_B} H_{ij} u_j - \sum_{n=1}^{N_V} M_{ij} \ddot{u}_j + \sum_{n=1}^{N_V} Q_{ij} \sigma_{jk} \]

The boundary element implementation transforms the system of integral equations to an equivalent algebraic system, which in matrix notation reads

\[ [G] \{p(t)\} - [H] \{u(t)\} - [M] \{\ddot{u}(t)\} + [Q] \{\sigma^p (t)\} = \{0\} \]

In eqn (4) matrices \([G]\) and \([H]\) correspond to the boundary integrals and \([M]\) and \([Q]\) to the inertial and initial stress domain integrals, respectively. The construction of the above matrices requires integrations in every element and cell as described in eqn (2). Regular and nearly singular integrals are evaluated by the standard Gaussian quadrature, while singular integrals by the Guiggiani & Gigante method [9].

The Houbolt scheme is selected for the integration in time because it gives excellent results with respect to stability and accuracy. Thus, velocity and acceleration are expressed in displacement terms as [10]

\[ \dot{u}_{n+1} = \frac{l}{6 \Delta t^2} \left( 11u_{n+1} - 18u_n + 9u_{n-1} - 2u_{n-2} \right) \]

\[ \ddot{u}_{n+1} = \frac{l}{\Delta t^2} \left( 2u_{n+1} - 5u_n + 4u_{n-1} - u_{n-2} \right) \]

If the current time step is the \((n+1)\), substitution of eqn (6) into eqn (4) gives

\[ \Delta t^2 [G] \{p^{n+1}\} - (\Delta t^2 [H] + 2[AM]) \{u_{n+1}\} = [M] \{-5u_n + 4u_{n-1} - u_{n-2}\} - \Delta t^2 [Q] \{\sigma_{n+1}\} \]

or in a compact form

\[ [G^*] \{p^{n+1}\} - [H^*] \{u_{n+1}\} = [M] \{-5u_n + 4u_{n-1} - u_{n-2}\} - [Q^*] \{\sigma_{n+1}\} \]

where

\[ [G^*] = \Delta t^2 [G], \ [H^*] = \Delta t^2 [H] + 2[AM], \ [Q^*] = \Delta t^2 [Q] \]
The assumption of zero initial displacement, velocity and initial stress gives the initial conditions

\[ [G]\{p_0\}+[M]\{\dot{u}_0\} = \{0\} \] (10)

Use of the boundary conditions enables one to solve the above equation and obtain the initial tractions and accelerations. After the application of the boundary conditions, eqn (8) becomes

\[ [GH]\begin{bmatrix} p \\ u \end{bmatrix}_{n+1} = [M]\{5u_{n} + 4u_{n-1} - u_{n-2}\} - [Q^*]\{\sigma_{n+1}\} + \{B\} \] (11)

where \([GH]\) corresponds to the \([G^*]\) and \([H^*]\) terms, while \(\{B\}\) arises from the known boundary conditions. The tractions and displacements in the left-hand side cannot be computed from eqn (11) because of the ignorance of the stress vector \(\{\sigma\}\) in the right hand side. The solution of this equation requires an iterative procedure. The computation of stresses via the way ‘‘displacements → strains → stresses’’ is selected because it is more efficient than the computation via integral equations, which requires much more computational effort [11]. The iterative scheme employed here is an extension of Banerjee’s static inelastic iterative algorithm [11] to dynamics. In every time step, starting with the assumption of an elastic structure, the solution of eqn (11) gives the first estimation for displacements and tractions. From these displacements, the strains are obtained and use of the constitutive equation described in the next section enable one to determine the elastic, plastic and total stresses. The plastic stresses in eqn (11) give a second estimation for displacements and tractions etc. If convergence is satisfied in this procedure, the next time step follows.

3 Constitutive relations

This paper covers materials with ductile behavior. Ductile materials are simulated successfully by the theory of plasticity. In this paper, the von Mises model [12,13] for elastic-perfectly plastic materials is adopted. This is ideal for the material description of various metals like steel, aluminum, and copper [12]. The inelasticity arises when the equivalent elastic stress \(\sigma_{eq}^e\) exceeds the yield stress \(\sigma_0\)

\[ \sigma_{eq}^e - \sigma_0 \geq 0 \] (12)

where

\[ \sigma_{eq} = \frac{1}{\sqrt{2}} \left[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \right]^{1/2} \] (13)
It is assumed that the strain and stress states are known from the previous time step \( t_A \). For the determination of the stress state at the present time step \( t_B \), one must define [13]

a) a first estimation of the total stress state

\[
\{\sigma_B\} = \{\sigma_A\} + [C]\{\Delta\varepsilon\}
\]  

(14)

where \([C]\) is elastic constitutive matrix and \(\{\Delta\varepsilon\}\) is the strain increment

b) the vector \(\{a_B\}\)

\[
\{a_B\} = \frac{1}{2\sigma_{eq,B}} \begin{bmatrix} 2\sigma_{x,B} - \sigma_{y,B} - \sigma_{z,B} \\ 2\sigma_{y,B} - \sigma_{z,B} - \sigma_{x,B} \\ 2\sigma_{z,B} - \sigma_{x,B} - \sigma_{y,B} \\ 6\tau_{xy,B} \\ 6\tau_{yz,B} \\ 6\tau_{zx,B} \end{bmatrix}
\]  

(15)

c) the parameter \( \Delta\lambda \)

\[
\Delta\lambda = \frac{\sigma_{eq,B} - \sigma_0}{\{a_B\}^T[C]\{a_B\}}
\]  

(16)

d) a second, improved estimation of the total stress state

\[
\{\sigma_{B1}\} = \{\sigma_B\} - \Delta\lambda[C]\{a_B\}
\]  

(17)

Convergence \( (\sigma_{eq} = \sigma_0) \) is obtained after a few iterations of steps b) to d). The computation of the total \( \{\sigma\} \) and the elastic \( \{\sigma^e\} \) stress tensors leads to the determination of the plastic stress tensor \( \{\sigma^p\} \) from

\[
\{\sigma^p\} = \{\sigma\} - \{\sigma^e\}
\]  

(18)

The plastic stress tensor is replaced directly in eqn (11) to compute the displacements and tractions of the structure.

4 Examples

This section describes two representative numerical examples in order to illustrate the use and demonstrate the advantages of the proposed three-dimensional dynamic elastoplastic boundary element method of analysis.
4.1 Dynamic analysis of a thick circular cylinder

Consider a thick circular cylinder of total length $L$ and inner and outer radii $r_i$ and $r_o$, respectively, made of an elastic-perfectly plastic von Mises material subjected to a suddenly applied uniform internal pressure $p$, as shown in Fig. 1. Due to symmetry of the problem, only one quarter of the cylinder can be analysed under either 2-D or 3-D conditions.

![Figure 1: Geometry and discretization of Example 1](image)

Fig. 1 shows also the discretization of the structure. The geometry and material parameters are $L = 150$ mm, $r_i = 100$ mm, $r_o = 200$ mm, $E = 210$ kN/mm$^2$, $v = 0.3$, $\rho = 7.85 \times 10^{-6}$ kg/mm$^3$, $\sigma_y = 355$ N/mm$^2$ (elastic-perfectly plastic behavior), and $p = 185$ N/mm$^2$. Fig. 2 shows the time history of the radial displacement at point A for elastic and elastoplastic material behavior as obtained by: (i) the proposed 3-D BEM scheme and (ii) the BEM/FEM 2-D scheme of Pavlatos & Beskos [14]. There is a very good agreement between the two methods.

![Figure 2: History of radial displacement at point A of Example 1](image)
4.2 Dynamic analysis of a simply supported beam

In this example, a simply supported beam subjected to a uniform impact loading is analyzed numerically by the proposed method using the von Mises model. The geometry and discretization of the structure appear in Fig. 3. Due to symmetry of the problem, only the half of the beam is investigated under either 2-D or 3-D conditions.

The geometry and material parameters are \( L = 24 \), \( H = 6 \), \( t = 2 \), \( E = 100 \), \( v = 0.333 \), \( \rho = 1.5 \), \( \sigma_y = 0.16 \) (elastic-perfectly plastic behavior) and \( \rho = 0.75\rho_o \) where \( \rho_o = 2\sigma_y H^2/L^2 \) is the static collapse load. Fig. 4 shows the time history of the vertical displacement at point A for elastic and elastoplastic material behavior as obtained by: (i) the proposed 3-D BEM scheme and (ii) the BEM/FEM 2-D scheme of Pavlatos & Beskos [14]. The agreement between the two methods is very good.
5 Conclusions

In this paper a BEM for transient dynamic elastoplastic analysis of 3-D solids and structures is presented. The method employs the static fundamental solution of the problem and this creates not only boundary integrals but also volume integrals as well due to the presence of inelasticity and inertial effects. Thus, boundary as well as domain (interior) elements are used in the space discretization of the problem. The implicit algorithm of Houbolt is employed for the numerical integration in time. Initial stress formulation and elastoplastic material behavior are used to simulate the inelasticity. The main aspects concerning the numerical implementation required for the solution of the nonlinear dynamic problem are also presented. Two numerical examples are described to illustrate the method and demonstrate its accuracy.

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References


