Transactions on Modelling and Simulation vol 28, © 2001 WIT Press, www.witpress.com, ISSN 1743-355X

Boundary integral equations for plane elastic problems posed on orientations of principal stresses and displacements

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Abstract

Boundary value problems (BVPs) of the plane elasticity posed in terms of the orientation of stresses and displacements are considered. These BVPs can be reduced to a system of homogeneous singular integral equations. Few equivalent systems are presented. They may posses a number of linearly independent solutions. The paper discusses the solvability of these equations and outlines the approach for seeking the number of solutions for the general case of 2D simple-connected domains bounded by smooth closed contours.

1. Introduction

A classical boundary value problem (BVP) of the plane elasticity requires one of the following surface conditions to be known on the entire boundary of a domain: (*i*) stress vector; (*ii*) displacement vector; or (*iii*) certain combinations of stress and displacement components (mixed problems). In all these cases the BVP is well posed and possesses a unique solution.

However there is a demand from different fields of Engineering and Geophysics to investigate BVPs of other (non-classical) types. There are attempts to model the elastic stress states of tectonic plates with different combinations of (*i*)-(*ii*) on different parts of plate margins, e.g., Coblentz et al, [1]. However due

to uncertain magnitudes of boundary stresses and displacements these attempts have failed to obtain a unique solution satisfying the observations of the principal stress orientations inside a tectonic plate. The fact that the solution is not unique can clearly be seen if, for instance, a constant spherical stress tensor was added to a solution. This would not violate the principal stress orientations and equilibrium while the boundary conditions would be changed. Since the number of solutions is *a-priory* unknown, the possibility to obtain a correct solution accidentally is quite low. Hence the approach based on the back analysis of the problems of such types should be revised. This revision can be done by considering the direct BVPs with unconventional types of boundary conditions.
For the example mentioned, the orientation of the boundary stresses and displacements obtained on the basis of geophysical observations can be used as the boundary conditions. This case is addressed in the present paper for a simply connected 2D domain bounded by a with smooth contour.

The first complete analysis of a plane elastic problem posed on orientations of stresses was made by Galybin and Mukhamediev [2]. The orientation of the principal axes of stress and the curvature of trajectories of the principal stress were used as the boundary conditions. They have been specified by the angle of the orientation of the major principal stress (with respect to a Cartesian frame) and its normal derivative on the domain boundary. It was found that solution of the problem is not unique. The problem may have no bounded solutions as well if the mentioned above angle gains a negative increment in anti-clockwise traversing the boundary. Otherwise the problem has a certain number of linear independent solutions. This number is determined by the number of complete revolutions of the principal axes of stress while traversing the contour. This fact distinguishes the problem considered from those classical ones formulated with the use of stress/displacement magnitudes where the solution is always unique.
It is evident that the determination of the number of solutions is a key question in the problem formulated by means of stress and displacement orientations. Any attempts to find a unique solution by using the methods of back analysis would fail until this number is known.

2. Formulation of the problem

2.1 General formulae

Let the following notations be introduced for the combinations of stresses and displacements in the Kolosov-Muskhelishvili formulae (e.g., Muskhelishvili, [3])

$$P = \frac{\sigma_{22} + \sigma_{11}}{2}, \quad D = \frac{\sigma_{22} - \sigma_{11}}{2} + i\sigma_{12}, \quad W = 2G(u + iv) \qquad (1)$$

Here σ_{ij} are stress components in a Cartesian coordinate system Ox_1x_2; P and D are spherical and deviatoric parts of the stress tensor respectively; $u+iv$ is the displacement vector; G is the shear modulus.

The general solution of a plane elastic problem can be expressed by means of two holomorphic functions φ(z) and ψ(z) of a complex variable $z=x_1+ix_2$ via the Kolosov-Muskhelishvili formulae

$$P=\varphi'(z)+\overline{\varphi'(z)},\quad D=\bar{z}\varphi''(z)+\psi'(z),\quad W=\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)}. \tag{2}$$

where κ=3-4ν for plane strain and κ=(3-ν)/(1+ν) for plane stress, ν is Poisson's ratio. It is seen from (2) that the stress function D is a bianalytic one (its second derivative with respect to the conjugated variable vanishes). It is also convenient to introduce another bianalytic function T associated with the function D as follows

$$\mathrm{T}(z,\bar{z})=\bar{z}\varphi'(z)+\psi(z) \tag{3}$$

Then the last two formulae in (2) can be presented in the form

$$D(z,\bar{z})=\mathrm{T}'_z(z,\bar{z}) \tag{4}$$

$$\overline{W(z,\bar{z})}=\kappa\overline{\varphi(z)}-\mathrm{T}(z,\bar{z}). \tag{5}$$

<u>Boundary values</u> Let Γ be a smooth closed contour bounding a simply connected domain X. On the boundary the variables z and $\bar{z}$ are not independent, hence the boundary value of a function of two variables $f(z,\bar{z})$ determined in X can be defined as follows

$$\lim_{z\to\zeta} f(z,\bar{z})=f(\zeta,\bar{\zeta})=f(\zeta),\quad \zeta\in\Gamma. \tag{6}$$

It should be noted that the same notation will be kept throughout the text for the function of two variables and its boundary value although the latter is the function of a single variable, ζ. Moreover the argument ζ at the boundary value of a function will in most cases be omitted.

2.2 Some forms for boundary conditions

The functions D and W can be presented in the complex form

$$D(z,\bar{z})=\sigma(z,\bar{z})e^{i\alpha(z,\bar{z})},\quad W(z,\bar{z})=\omega(z,\bar{z})e^{i\beta(z,\bar{z})} \tag{7}$$

Let the direction of displacements and stresses be known on Γ, then the boundary conditions can be expressed in the form

$$\Gamma:\quad \arg D=\alpha,\quad \arg W=\beta. \tag{8}$$

Here α=α(ζ) and β=β(ζ) are given real-valued functions. They satisfy the Hölder conditions on Γ except perhaps a number of points where they may have the discontinuities of the first kind and suffer the jump of 2π, which is the consequence of the determination of *arg*.

Expressing the fact that s and w are real-valued functions one can rewrite the boundary conditions given by formulae (8) thorough the boundary values of complex-valued functions T and $D = T'_z$ as follows

$$\operatorname{Im}\left[e^{-i\alpha}T'_z\right] = 0, \quad \operatorname{Im}\left[e^{i\beta}\left(\kappa\overline{\varphi} - T\right)\right] = 0 \tag{9}$$

The latter represent the boundary value problem for the determination of functions φ and T. This problem can also be written down in the form of a single complex equation. It is convenient to represent two scalar conditions in a complex form. For this purpose one differentiates the second condition with respect to the arc length, *s*.

$$f'_s = e^{i\theta} f'_z + e^{-i\theta} f'_{\bar{z}} \tag{10}$$

where θ is the angle between the tangent to a point of the contour Γ and the real axis, Imz=0 which can also be found as follows

$$e^{-2i\theta} = \frac{d\overline{\zeta}}{d\zeta}, \quad \zeta \in \Gamma \tag{11}$$

Since the derivative of a real valued function with respect to *s* remains real, the differentiation of the second boundary conditions in (9) leads to the following expression

$$\operatorname{Im}\left[e^{i\beta}\left(\kappa\overline{\varphi} - T\right)'_s\right] = -e^{i\beta}\beta'_s\left(\kappa\overline{\varphi} - T\right) \tag{12}$$

According (10) the derivatives have the form

$$\overline{\varphi'_s} = e^{-i\theta}\overline{\varphi}', \quad T'_s = e^{i\theta}T'_z + e^{-i\theta}\varphi' \tag{13}$$

Representing (9) in the form

$$\begin{aligned} &e^{-i\alpha}T'_z - e^{i\alpha}\overline{T'_z} = 0 \\ &e^{i\beta}\left(\kappa\overline{\varphi} - T\right)'_s - e^{-i\beta}\left(\kappa\varphi - \overline{T}\right)'_s = -2ie^{i\beta}\beta'_s\left(\kappa\overline{\varphi} - T\right) \end{aligned} \tag{14}$$

and using (13) one obtains

$$\begin{aligned} &e^{-i\alpha}T'_z - e^{i\alpha}\overline{T'_z} = 0 \\ &e^{-i(\beta+\theta)}\overline{T'_z} - e^{i(\beta+\theta)}T' - 2i\operatorname{Im}\left\{\left[e^{i(\beta-\theta)} + \kappa e^{-i(\beta-\theta)}\right]\varphi'\right\} = -2ie^{i\beta}\beta'_s\left(\kappa\overline{\varphi} - T\right) \end{aligned} \tag{15}$$

Multiplying the first equation in (15) by $-e^{-i(\beta+\theta-\alpha)}$ and summing with the second equation one obtains the boundary condition in the following complex form

$$e^{-i\alpha}\sin(\alpha + \beta + \theta)T'_z + \operatorname{Im}\left\{\left[e^{i(\beta-\theta)} + \kappa e^{-i(\beta-\theta)}\right]\varphi'\right\} = e^{i\beta}\beta'_s\left(\kappa\overline{\varphi} - T\right) \tag{16}$$

It should be noted that (16) is not fully equivalent to (9) because of the differentiation used.

3. Singular integral equations

3.1 Integral representations for holomorphic functions.

Let the holomorphic functions be presented through the Cauchy integrals

$$\varphi(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{g(t)}{t-z}dt,\quad \psi(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{h(t)-\bar{t}g(t)}{t-z}dt \tag{17}$$

where the unknown complex valued functions $g=g(t)$ and $h=h(t)$ satisfy the Hölder condition. These functions are not independent from each other due to the fact that two (scalar) conditions are needed to determine both holomorphic functions. For this reason the term $\bar{t}g(t)$ is introduced in expression for $\psi(z)$ for the sake of convenience. This arbitrariness is used further in order to obtain various form of singular integral equation for the problem considered.

The following is valid for the (k)-derivative of the holomorphic functions (e.g., Muskhelishvili [3])

$$\varphi^{(k)}(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{g^{(k)}(t)}{t-z}dt \tag{18}$$

In view of (17) the function T has the following representation

$$\mathrm{T}(z,\bar{z})=\frac{1}{2\pi i}\int_{\Gamma}\frac{h(t)}{t-z}dt-\frac{1}{2\pi i}\int_{\Gamma}\frac{\bar{t}-\bar{z}}{t-z}g'(t)\,dt \tag{19}$$

For derivative of T with respect to z one can obtain the following expression

$$\mathrm{T}_z'(z,\bar{z})=\frac{1}{2\pi i}\int_{\Gamma}\frac{h'(t)-e^{-2i\theta}g'(t)}{t-z}dt+\frac{1}{2\pi i}\int_{\Gamma}\left[e^{-2i\theta}-\frac{\bar{t}-\bar{z}}{t-z}\right]\frac{g'(t)}{t-z}dt \tag{20}$$

These representations are valid for any point of interior as well as exterior domains of the entire complex plane divided by for the contour Γ. Therefore the problem discussed further can be related to both these domains with the only difference being that after limiting transition to the contour all the functions in formulae (17)-(20) have different values depending upon which side this transition is made. Therefore all these functions suffer certain jumps across the contour Γ. This can be determined from solving a corresponding system of singular integral equations as discussed in the next Subsection.

3.2 Boundary integral equations

Boundary values of the functions entering into the boundary conditions can be found by the Sokhotski-Plemelj formulae (e.g., [3]). Here they are specified for the interior domain

$$
\begin{aligned}
&2\varphi = g + \mathbf{I}(g), \quad 2\varphi' = g' + \mathbf{I}(g'), \\
&2\overline{\varphi} = \overline{g} - \mathbf{I}(\overline{g}) + \mathbf{R}_1(\overline{g}) \\
&2\mathrm{T} = h + \mathbf{I}(h) + \mathbf{R}_2(g) \\
&\mathrm{T}_z' = h' - e^{-2i\theta} g' + \mathbf{I}\left(h' - e^{-2i\theta} g'\right) + \mathbf{R}_2(g')
\end{aligned}
\tag{21}
$$

where $\mathbf{I}(..)$ is singular and $\mathbf{R}_1(..)$, $\mathbf{R}_2(..)$ are regular operators. They have been introduced as follows ($\zeta \in \Gamma$)

$$
\begin{aligned}
&\mathbf{I}(g) = \frac{1}{\pi i} \int_\Gamma \frac{g(t)}{t-\zeta} dt \\
&\mathbf{R}_1(\overline{g}) = \frac{1}{\pi i} \int_\Gamma \left[1 - \frac{t-\zeta}{\overline{t}-\overline{\zeta}} e^{-2i\theta(t)}\right] \frac{\overline{g(t)}}{t-\zeta} dt \\
&\mathbf{R}_2(g) = \frac{1}{2\pi i} \int_\Gamma \left[e^{-2i\theta} - \frac{\overline{t}-\overline{\zeta}}{t-\zeta}\right] \frac{g(t)}{t-\zeta} dt
\end{aligned}
\tag{22}
$$

The singular operator satisfies the following properties

$$
g = \mathbf{I}^2(g), \quad \overline{\mathbf{I}(g)} = -\mathbf{I}(\overline{g}) + \mathbf{R}_1(\overline{g}) \tag{23}
$$

Substitution of (21) into boundary conditions in the form (9) leads to

$$
\begin{cases}
\mathrm{Im}\left\{e^{-i\alpha}\left[h' - e^{-2i\theta} g' + \mathbf{I}\left(h' - e^{-2i\theta} g'\right) + \mathbf{R}_2(g')\right]\right\} = 0 \\
\mathrm{Im}\left\{e^{i\beta}\left[\kappa \overline{g} - h - \kappa \mathbf{I}(\overline{g}) - \mathbf{I}(h) + \kappa \mathbf{R}_1(\overline{g}) - \mathbf{R}_2(g)\right]\right\} = 0
\end{cases}
\tag{24}
$$

In order to derive different forms of singular integral equations, the arbitrariness in the choice of the functions g and h can be used. It should be noted that somewhat similar approach has been used by Linkov [4] for the analysis of the classical boundary value problems of plane elasticity. The cases $h = \overline{g}$ and $h = \kappa \overline{g}$ considered below correspond to the well known representation used by Sherman (e.g., Muskhelishvili, [3]) in the first and second boundary value problem receptively.

Case 1 provides the continuity of the stresses across the contour. This assumption does not lead to the loss of generality. It actually provides two solutions for interior and exterior domains having the same boundary Γ.

Assuming $h = \bar{g}$ and taking into account that

$$h' = e^{-2i\theta}\bar{g} \tag{25}$$

one obtains the following system of singular integral equations

$$\begin{cases} \operatorname{Im}\left\{e^{-i\alpha}\left[e^{-2i\theta}(\bar{g}-g') + \mathbf{I}\left(e^{-2i\theta}(\bar{g}-g')\right) + \mathbf{R}_2(g')\right]\right\} = 0 \\ \operatorname{Im}\left\{e^{i\beta}\left[(\kappa+1)\bar{g} - (\kappa+1)\mathbf{I}(\bar{g}) + \kappa\mathbf{R}_1(\bar{g}) - \mathbf{R}_2(g)\right]\right\} = 0 \end{cases} \tag{26}$$

In this case both equations have complete dominant part (terms left after neglecting the regular operators).

Case 2 requires the continuity of the displacement across the contour. Assuming $h = \kappa\bar{g}$ and taking (25) into account one has

$$\begin{cases} \operatorname{Im}\left\{e^{-i\alpha}\left[e^{-2i\theta}(\kappa\bar{g}-g') + \mathbf{I}\left(e^{-2i\theta}(\kappa\bar{g}-g')\right) + \mathbf{R}_2(g')\right]\right\} = 0 \\ \operatorname{Im}\left\{e^{i\beta}\left[2\kappa\mathbf{I}(\bar{g}) - \kappa\mathbf{R}_1(\bar{g}) + \mathbf{R}_2(g)\right]\right\} = 0 \end{cases} \tag{27}$$

In this case second equation has no non-integral term in its dominant part.

Case 3 requires the absence of non-integral terms on both equations. This can be achieved if functions g and h satisfy the following system

$$\operatorname{Im}\left[e^{-i\alpha}\left(h' - e^{-2i\theta}g'\right)\right] = 0, \quad \operatorname{Im}\left[e^{i\beta}(\kappa\bar{g} - h)\right] = 0 \tag{28}$$

Then the system becomes

$$\begin{cases} \mathbf{I}\left(h' - e^{-2i\theta}g'\right) + \mathbf{R}_3(h', g') = 0 \\ e^{i\beta}\mathbf{I}(\kappa\bar{g}) + e^{-i\beta}\mathbf{I}(\bar{h}) + \mathbf{R}_3(\bar{h}, g) = 0 \end{cases} \tag{29}$$

where regular operators have the following form

$$\begin{aligned} \mathbf{R}_3(h', g') &= \frac{e^{i\alpha}}{2}\left[\mathbf{R}_{2\alpha}\left(h' - e^{-2i\theta}g'\right) - \mathbf{R}_1\left(\overline{h' - e^{-2i\theta}g'}\right)\right] - i\operatorname{Im}\left[e^{-i\alpha}\mathbf{R}_2(g')\right] \\ \mathbf{R}_3(\bar{h}, g) &= \frac{e^{-i\beta}}{2}\left[\mathbf{R}_{2\beta}\left(\kappa g + \bar{h}\right) - \mathbf{R}_1\left(\kappa g + \bar{h}\right)\right] - i\operatorname{Im}\left\{e^{i\beta}\left[\kappa\mathbf{R}_1(\bar{g}) - \mathbf{R}_2(g)\right]\right\} \\ \mathbf{R}_{2\alpha}\left(h' - e^{-2i\theta}g'\right) &= \mathbf{I}\left(e^{-2i\alpha}\left(h' - e^{-2i\theta}g'\right)\right) - e^{-2i\alpha}\mathbf{I}\left(h' - e^{-2i\theta}g'\right) \\ \mathbf{R}_{2\beta}\left(\kappa g + \bar{h}\right) &= \mathbf{I}\left(e^{2i\beta}(\kappa\bar{g} - h)\right) - e^{2i\beta}\mathbf{I}(\kappa\bar{g} - h) \\ \mathbf{R}_{2\beta}\left(\kappa g + \bar{h}\right) &= \mathbf{I}\left(e^{2i\beta}(\kappa\bar{g} - h)\right) - e^{2i\beta}\mathbf{I}(\kappa\bar{g} - h) \end{aligned} \tag{30}$$

It should be noted that equations (28) and (29) have to be solved simultaneously.

Case 4 can be obtained by substitution of (21) into boundary condition in the complex form, which leads to

Transactions on Modelling and Simulation vol 28, © 2001 WIT Press, www.witpress.com, ISSN 1743-355X

$$\begin{aligned}&\mathrm{Im}\left(Ag'\right)-e^{-2i\theta}g'+Bh'-e^{i\beta}\beta_s'\left(\kappa\overline{g}-h\right)+\\&\quad+\mathrm{Im}\left(A\mathbf{I}\left(g'\right)\right)+\mathbf{I}\left(h'-e^{-2i\theta}g'\right)+e^{i\beta}\beta_s'\mathbf{I}\left(\kappa\overline{g}+h\right)+\mathbf{R}\left(g\right)=0\end{aligned}\tag{31}$$

where the following notations have been used

$$\begin{aligned}&A=e^{i(\beta-\theta)}+\kappa e^{-i(\beta-\theta)},\quad B=e^{-i\alpha}\sin\left(\alpha+\beta+\theta\right),\\&\mathbf{R}\left(g\right)=\mathbf{R}_2\left(g'\right)-e^{i\beta}\beta_s'\left(\kappa\mathbf{R}_1\left(\overline{g}\right)-\mathbf{R}_2\left(g\right)\right)\end{aligned}\tag{32}$$

Assuming similar relationships between g and h as in cases 1-3 one can derive corresponding complex forms for these cases.

The boundary integral equations obtained in the this Subsection are homogeneous equations. Their solvability can be established on the basis of the Noether's theorems (e.g., Gakhov [6]). In particular, for each of these equations the number of its linearly independent solutions cannot be less than the corresponding number of solutions of the dominant equation (obtained from the complete equation by neglecting the regular part). The analysis for determination of the number of linearly independent solutions can also be based on the statement that the cases in which the numbers of solutions of the complete and the dominant equations are different from each other should be considered as the exceptional ones Gakhov [6]. This result can be derived by applying the Carleman-Vekua method for the regularisation of the complete singular integral equation.

In particular, the number of solution of the first equation in (26) has been determined by Galybin and Mukhamediev [2] by reduction of the dominant part of this equation to the corresponding Riemann boundary value problems. Then the number of solutions is determined by the positive integer number, N, determined as follows

$$\mathrm{N}=\frac{1}{\pi i}\left\langle \ln e^{i(2\theta+\alpha)}\right\rangle_\Gamma=\frac{1}{\pi}\int_\Gamma\left(2\theta'(t)+\alpha'(t)\right)dt\tag{33}$$

Here symbol $\langle..\rangle_\Gamma$ denotes the increment of the function in brackets while over the contour Γ. It is shown that the equation has N-1 independent solutions; if N is then no solutions may exist in the class of the function bounded inside the domain.

The solvability of the second equation in (26) can also be investigated by considering its reduction of the dominant part. By taking into account the general homogeneous solution of the first (dominant) equation the second equation can be transformed to a non-homogeneous form. In this case the right hand side would contain up to N complex constant entering into the solution for Im(g'). This obviously will not induce any additional solution of the first equation if the first dominant equation is solvable. In this case the general number of solutions would be the sum of solutions of the first and second dominant equations. On the other hand the first dominant equations may have no solutions for the homogeneous right-hand side. In this case a number of additional conditions will be imposed on the right-hand side. This number also depends on the index of the first dominant equations and should be less than N to provide the solvability of the complete system.

This outlines the approach to investigation of the solvability of the boundary value problem formulated in terms of displacement and stress orientations on the boundary of the 2D domain. The general case of orientations should be investigated in detail. However this lies behind the aims of the present paper.

4. Conclusion

A number of equivalent system of singular integral equations have been derived for solving the boundary value problems of the plane elasticity posed in terms of the orientation of stresses and displacements on the closed contour. It is shown that these equations may have non-unique solutions. The solvability of these equations can be investigated on the basis of analysis of solutions of the dominant integral equations obtained from the complete systems by omitting the regular integral operators. This approach may provide an effective estimation of the number of linearly independent solution of the BVP considered for the general case of a simple-connected domain.

Acknowledgment. This work is supported by the Australian Research Council (large grant A69941059).

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