Interaction between elliptic hole and crack in thin plate under uniform bending heat flux

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Abstract

The interaction problem between an elliptic hole and a crack in a thin plate under uniform bending heat flux is considered. For solving the interaction problem, a solution for an infinite thin plate with an adiabatic elliptic hole under uniform bending heat flux, and two Green's functions of the plate under a bending heat source couple and a bending dislocation are given. The interaction problem then reduces into singular integral equations using the Green's functions and the principle of superposition. After the equations are solved numerically, the moment intensity factors at crack tips are evaluated.

1 Introduction

The fracture toughness of engineering brittle materials is influenced significantly by the microstructure of the material, such as micro-hole, inclusion or micro-crack. Thus the interaction problems between micro-defect and macro-crack or among the micro-defects have received considerable attentions. The interaction problems for isotropic materials were studied by Hoagland [1], Gross [2], Horri and Nemat-Nasser [3], Kachanov [4] and Rubinstein [5]. The same problem under anti-plane load were researched by Gong [6] and Chen [7]. The interaction problems for anisotropic materials have been shown by Ma et al. [8], Qin [9] and Han [10]. This kind of problems for interface cracks were considered by Zhao and Chen [11]. However, unlike in the case of elasticity, relatively little work has been done for the analysis of interacting micro-defects of thermoelasticity.

This paper aims at solving an interaction problem between an adiabatic
elliptic hole and an adiabatic crack in an infinite thin plate under uniform bending heat flux. Firstly, the solution for the elliptic hole under uniform bending heat flux, and two Green’s functions of the plate under a bending heat source couple and a bending dislocation are derived. Then, the interaction problem is reduced into two systems of singular integral equation using these Green’s functions and the principle of superposition. After the equations have been solved numerically, the moment intensity factors at crack tips are evaluated.

2 Basic formulations

Consider an adiabatic elliptic hole in an infinite thin plate of thickness \( h \) subjected to bending temperature field, in which the temperature varies linearly and anti-symmetric to the mid-plane of the plate. A rectangular coordinates system \( Ox\bar{y}\bar{z} \) is set upon in a way that the \( Ox \) and the \( Oy \) axes locate in the mid-plane, while the \( Oz \)-axis vertically downwards. The following mapping function is used to map the exterior of an elliptic hole with \( a \) and \( b \) being the \( x \) and \( y \) semi-axes in the \( z \)-plane onto an exterior of a unit circle in the \( \zeta \)-plane,

\[
z = \alpha(\zeta) = E_0\zeta + \frac{E_1}{\zeta}, \quad E_0 = \frac{a+b}{2}, \quad E_1 = \frac{a-b}{2}
\]

In the \( \zeta \)-plane, temperature gradient \( T \) on the \( z \)-axis can be expressed in term of an analytic function \( Y(\zeta) \) as [12]

\[
T(\zeta, \bar{\zeta}) = \frac{1}{2} \left( Y(\zeta) + Y(\bar{\zeta}) \right),
\]

and the heat flux gradients can be given by

\[
q_x - iq_y = -k \frac{Y'(\zeta)}{\omega'(\zeta)},
\]

where \( q_x \) and \( q_y \) represent the heat flux gradients in the \( x \) and \( y \) directions, respectively, and \( k \) denotes the thermal conductivity of the material. Moreover, the heat flux boundary condition on the unit circle can be given by

\[
-k(Y(\sigma) - Y(\bar{\sigma})) = 2i \int_{\sigma} q_n ds + \text{const.}
\]

where \( \sigma \) denotes a value of \( \zeta \) on the boundary of the unit circle, the integration is carried out along the boundary, and \( q_n \) represents the normal heat flux gradient.

The bending moments, twisting moments and shear forces can be expressed by two moment functions, \( \varphi(\zeta) \) and \( \psi(\zeta) \) [13]

\[
M_x - M_y + 2iH_y = \frac{2D(1-\nu)}{\omega'(\zeta)} \left[ \frac{\varphi'(\zeta)}{\omega'(\zeta)} + \psi'(\zeta) \right],
\]

\[
M_x + M_y = -4D(1+\nu)Re \left[ \frac{\varphi'(\zeta)}{\omega'(\zeta)} + \frac{1}{2} \alpha Y'(\zeta) \right],
\]

\[
N_x - N_y = -\frac{4D}{\omega'(\zeta)} \left[ \frac{\varphi'(\zeta)}{\omega'(\zeta)} + \frac{1}{4} (1+\nu) \alpha Y'(\zeta) \right],
\]

where \( \alpha \) represents the linear thermal expansion coefficient, and \( D \) denotes the
Flexural rigidity of the plate, which can be expressed in terms of the modulus of elasticity $E$ and Poisson's ratio $\nu$ as $D = Eh^3/[12(1-\nu^2)]$. The resultant force in the $z$-axis direction, $P^*$, and resultant moments, $M_x^*$ and $M_y^*$, of the force applied to any contour $L$ can be obtained from eqn (5)

$$P^* = 2iD \left[ \frac{\phi'(\zeta)}{\omega'(\zeta)} - \frac{\phi'(\zeta)}{\omega'(\zeta)} + \frac{(1+\nu)\alpha}{4} \{Y(\zeta) - \overline{Y(\zeta)}\} \right]_L$$

$$M_x^* + iM_y^* + \nu \omega(\zeta)P^* = -D(1-\nu) \left[ \kappa \frac{\omega(\zeta)}{\omega'(\zeta)} \frac{\phi'(\zeta)}{\phi'(\zeta)} - \psi(\zeta) + \frac{1+\nu}{1-\nu} \alpha \int \{Y(\zeta)\omega'(\zeta)\} d\zeta \right]_L$$

The displacement and moment boundary conditions can be expressed as

$$\phi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)} \phi'(\sigma) + \psi(\sigma) = \frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y}$$

$$\kappa \phi(\sigma) - \frac{\omega(\sigma)}{\omega'(\sigma)} \phi'(\sigma) - \psi(\sigma) + \frac{1+\nu}{1-\nu} \alpha \int Y(\sigma) \omega'(\sigma) d\sigma = -\frac{1}{D(1-\nu)} \int [M(s) + i \int p(s) ds] (dx + idy)$$

where $M(s)$ and $P(s)$ denote the bending moment and bending force applied to the boundary [13].

3 Fundamental solutions

Consider now three fundamental solutions for an adiabatic elliptic hole in an infinite thin plate subjected to three different thermal loading conditions, i.e. uniform bending heat flux, a bending heat source couple, and a bending dislocation.

3.1 Elliptic hole under uniform bending heat flux

The adiabatic elliptic hole in an infinite thin plate is subjected to uniform bending heat flux in which the heat flux gradient against thickness is a constant value, $q e^{-\delta}$. The angle, $\delta$, is measured from the $x$-axis to the direction of the heat flux (see Fig.2).

The temperature function $Y_4(\zeta)$ can be decomposed into two parts

$$Y_4(\zeta) = Y_1(\zeta) + Y_2(\zeta)$$

where $Y_1(\zeta)$ denotes the temperature function for the plate in absence of the hole. From eqn (3), it can be derived as

$$Y_1(\zeta) = -\frac{q}{k} e^{-\alpha} \omega(\zeta)$$

The function $Y_2(\zeta)$ is a holomorphic function. Substituting eqns (9) and (10) into eqn (4) and taking the adiabatic condition into consideration, then multiplying by a factor $d\sigma/(\sigma-\zeta)$, the function $Y_2(\zeta)$ can be derived by carrying out the Cauchy integration along the unit circle. Finally the temperature function is obtained as
The moment functions for the problem can also be decomposed into two parts

\[ \varphi_q(\zeta) = \varphi_1(\zeta) + \varphi_2(\zeta), \]
\[ \psi_q(\zeta) = \psi_1(\zeta) + \psi_2(\zeta) \]

(12)

\( \varphi_1(\zeta) \) and \( \psi_1(\zeta) \) are multiple-value valuedness conditions of resultant moment eqn (8) and displacement eqn (7) around the hole

\[ \varphi_1(\zeta) = A \log(\zeta) \]
\[ \psi_1(\zeta) = \bar{A} \log(\zeta) \]

(13)

where

\[ A = \frac{(1 + \nu)aqE_0}{4k} (E_0 e^{i\delta} - E_1 e^{-i\delta}) \]

(14)

\( \varphi_2(\zeta) \) and \( \psi_2(\zeta) \) are two holomorphic functions. Inserting eqn (12) into eqn (8), the following equation is obtained form the moment-free condition along the hole edge

\[ \kappa \varphi_2(\sigma) - \frac{\omega(\sigma)}{\omega'(\sigma)} \varphi_1(\sigma) - \psi_2(\sigma) = -\frac{1+\nu}{1-\nu} \alpha Y_q(\sigma) \omega'(\sigma) d\sigma \]

(15)

Substituting eqn (13) into eqn (15) and Multiplying both sides by \( d\sigma(\sigma - \zeta) \) and carrying out the Cauchy integration along the unit circle, the function \( \varphi_2(\zeta) \) is obtained

\[ \varphi_2(\zeta) = \frac{(1 + \nu)(1 + \kappa)aqE_0E_1}{8k \kappa} e^{i\delta} \frac{1}{\zeta^2} \]

(16)

Multiplying the conjugate form of eqn (15) by \( d\sigma(\sigma - \zeta) \) and carrying out the Cauchy integration along the unit circle, the function \( \psi_2(\zeta) \) is obtained

\[ \psi_2(\zeta) = -\frac{(1 + \nu)(1 + \kappa)aqE_0}{4k} \left[ \frac{E_0 e^{i\delta}}{2\zeta^2} + \frac{\omega(1/\zeta)}{\omega'(\zeta)} \left( \frac{E_0 e^{i\delta} - E_1 e^{-i\delta}}{1+\kappa} \frac{1}{\zeta^2} - \frac{E_1 e^{i\delta}}{\kappa \zeta^3} \right) \right] \]

(17)

3.2 Green's function of bending heat source couple

Consider the problem of an adiabatic elliptic hole under a bending heat source couple. The heat intensity of the couple varies linearly against the thickness and anti-symmetric about the mid-plane of the plate. Direction of the couple is denoted by \( \beta \) and is measured from the \( x \)-axis to the direction of the heat flux near the couple (see Fig. 1).

The temperature gradient function of this problem can be obtained from superposition of temperature functions for bending heat source and sink. The temperature function for bending heat source with heat intensity gradient \( M \) can
be given by
\[ Y_\nu(\zeta') = -\frac{M}{2\pi k} \left[ \log(\zeta' - \zeta_\nu) + \log(\zeta - \zeta_\nu) \right] \tag{18} \]
where \( \zeta_\nu \) denotes the coordinate on the \( \zeta \)-plane of the bending heat source point, and \( \zeta_\nu = 1/\zeta_\nu \). Similarly the temperature function for bending heat sink at \( \zeta_\nu \) can be given
\[ Y_s(\zeta') = \frac{M}{2\pi k} \left[ \log(\zeta' - \zeta_s) + \log(\zeta - \zeta_s) \right] \tag{19} \]
Setting the pair heat source and sink move to a point along the direction \( \beta \) (\( \text{dr} \to 0 \), see Fig.1), the temperature function for the heat source couple can be obtained
\[ Y_\sigma(\zeta) = \frac{R}{2\pi k} \left[ \frac{e^{i\beta}}{\omega'(\zeta_\nu)(\zeta - \zeta_\nu)} - \frac{e^{-i\beta} \zeta_\nu^2}{\omega'(\zeta_\nu)(\zeta - \zeta_\nu)} \right] \tag{20} \]
where \( R \) denotes the distribution density of the bending heat source couple and \( R = M \text{dr} \) (see Fig.1).

Figure 1: The pair heat source and sink move to a point along the angle \( \beta \).

Following the same procedures described in section 3.1, the moment functions for heat source couple can be derived as
\[
\begin{align*}
\varphi_\nu(\zeta') &= \varphi_1(\zeta') + \varphi_2(\zeta'), \\
\psi_\nu(\zeta') &= \psi_1(\zeta') + \psi_2(\zeta'), \\
\varphi_1(\zeta') &= -\frac{\alpha R(1+\upsilon)}{8\pi k} \left[ e^{i\beta} \log(\zeta' - \zeta_\nu) + A \log(\zeta') \right] \\
\psi_1(\zeta') &= -\frac{\alpha R(1+\upsilon)}{8\pi k} \left[ e^{-i\beta} \log(\zeta' - \zeta_\nu) - \frac{\omega(\zeta_\nu)}{\omega'(\zeta_\nu)} e^{i\beta} + A \log(\zeta') \right] \\
A &= \frac{E_1 e^{i\beta}}{\omega(\zeta_\nu) \zeta_\nu^2} - \frac{E_0 e^{-i\beta}}{\omega'(\zeta_\nu) \zeta_\nu^2} \\
\varphi_2(\zeta') &= -\frac{\alpha R(1+\upsilon)(1+\kappa)}{8\pi k \kappa} \left[ \log(\zeta' - \zeta_\nu) \left\{ \frac{e^{-i\beta} \zeta_\nu^2}{\omega(\zeta_\nu)} - \frac{e^{i\beta}}{1+\kappa} \right\} + \frac{E_1}{\zeta} \left\{ \omega'(\zeta_\nu) - \omega'(\zeta_\nu) \right\} + \frac{\omega(\zeta_\nu') - \omega(\zeta_\nu)}{\omega'(\zeta_\nu)(\zeta - \zeta_\nu')(1+\kappa)} \right] \\
&\quad + \frac{e^{i\beta}}{\zeta} \left\{ \frac{e^{-i\beta} \zeta_\nu^2}{\omega'(\zeta_\nu)} - \frac{e^{i\beta}}{1+\kappa} \right\} + \frac{\omega(\zeta_\nu') - \omega(\zeta_\nu)}{\omega'(\zeta_\nu)(\zeta - \zeta_\nu')(1+\kappa)} \right] \\
\end{align*}
\]
3.3 Green’s function of bending dislocation

The problem of moment-free elliptic hole in an infinite thin plate under a bending dislocation at point $\zeta_a$ has been given by Wang and Hasebe [14], whose solutions are given by

$$\varphi_D(\zeta) = \varphi_1(\zeta) + \varphi_2(\zeta),$$

$$\psi_D(\zeta) = \psi_1(\zeta) + \psi_2(\zeta)$$

where

$$\varphi_1(\zeta) = \frac{L}{2\pi(1 + \kappa)} \log(\zeta - \zeta_a)$$

$$\psi_1(\zeta) = \frac{1}{2\pi(1 + \kappa)} \left[ L \kappa \log(\zeta - \zeta_a) - \frac{L \omega(\zeta_a)}{\omega(\zeta_a)} \right]$$

$$\varphi_2(\zeta) = -\frac{1}{2\pi(1 + \kappa)} \left[ L \log(\zeta - \zeta_a) - \frac{L \omega(\zeta_a)}{\kappa \omega(\zeta_a)} \left( \omega(\zeta_a) - \omega(\zeta) \right) \right]$$

$$\psi_2(\zeta) = -\frac{\omega(\zeta)}{\omega(\zeta)} \varphi(\zeta) + \frac{1}{2\pi(1 + \kappa)} \left[ \omega(\zeta_a) \frac{L}{\omega(\zeta_a)} \frac{\log(\zeta - \zeta_a)}{\zeta} \right]$$

where the bending dislocation coefficient $L$ is defined as

$$L = \left\{ \frac{\partial w}{\partial x} + i \frac{\partial v}{\partial y} \right\}_c$$

in which $w$ denotes the deflection of the mid-plane of the plate, and $\{ \}_c$ means the increment of the embraced expression when moving around the dislocation point in the clockwise direction.

4 Singular integral equations

The original interaction problem shown in Fig.2 in which an adiabatic elliptic hole and an arbitrarily located and orientated adiabatic crack with length $2c$ in an infinite thin plate are subjected to uniform bending heat flux, can reduced to

Figure 2: An adiabatic elliptic hole interacts with an adiabatic crack.
superposition of three subproblems. One is the problem of the elliptic hole under uniform bending heat flux whose solutions has been obtained in section 3.1. The second is the problem of the elliptic hole under distributed bending heat source couples, \( R(s) \), at the location of the crack whose solution can be derived by integrating the Green’s function of heat source couple obtained in section 3.2. The third problem is that of the elliptic hole under distributed bending dislocations, \( D,(s) \), at the location of the crack. The solution for the third problem can be obtained by integrating the Green’s function of bending dislocation listed in section 3.3.

From the adiabatic condition along the crack face and eqn (4), the following singular integral equation with the Cauchy kernel is derived

\[
\text{Im} \left[ \int_c R(t) \left( Y_G(t,s) - Y_G(t,s_0) \right) dt \right] = - \text{Im} \left[ Y_q(t,s) - Y_q(t,s_0) \right]
\]

\(-c \leq s \leq c\)

in which \( s_0 \) is an arbitrary point on the crack face. The unknown heat source couple distribution function, \( R(t) \), can be further reduced to the following form

\[
R(t) = \sqrt{c^2 - t^2} r(t)
\]

where \( r(t) \) denotes a unknown function. By standard integral methods, the unknown function \( r(t) \) can be obtained numerically [15, 16].

From the moment-free condition along the crack face, we can know that the normal moment component \( M_n(s) \) and the sum of twisting moments \( H_m(s) \) and resultant shear force \( P_r(s) \) at any point \( s \) on the crack face should be zero. Thus a singular integral equation with the Cauchy kernel is reduced

\[
\int_c D,(t) \left[ M_n^D(t,s) + i \left( H_m^D(t,s) + P_r^D(t,s) \right) \right] dt =

-M_n^D(s) + M_n^R(s) - i \left( H_m^D(s) + H_m^R(s) + P_r^D(s) + P_r^R(s) \right)
\]

where \( M_n^D(t,s) \), \( H_m^D(t,s) \) and \( P_r^D(t,s) \) denote the bending moment component, twisting moment component and resultant shear force at a point \( s \) on the crack face induced by a bending dislocation at another point \( t \) on the crack. The three components can be easily calculated from eqns (5), (6) and (26). \( M_n^D(s) \), \( H_m^D(s) \), \( P_r^D(s) \), \( M_n^R(s) \), \( H_m^R(s) \) and \( P_r^R(s) \) in eqn (33) denote the bending moment component, twisting moment component and resultant shear force at the point \( s \) induced from the first and the second subproblems, respectively. They can be calculated from eqns (5), (6), (12) and (21). From the displacement single-valuedness condition on the crack face, the following integral equation is necessary

\[
\int_c D,(t) dt = 0
\]

Moreover, the dislocation distribution function for a crack problem can be further assumed as

\[
D,(t) = d,(t) \sqrt{c^2 - t^2}
\]

where \( d,(t) \) is a unknown function. Inserting eqn (35) into eqns (33) and (34), the singular integral equations can be solved using the standard numerical integral method [15, 16]. Then, the bending moment and twisting moment intensity
factors can be given by
\[ k_b + ik_s = \frac{D(3 + \nu)e^{-i\phi}}{(1 + \kappa)s} d(c) \] (36)
for the right crack tip and
\[ k_b + ik_s = \frac{-D(3 + \nu)e^{-i\phi}}{(1 + \kappa)s} d(-c) \] (37)
for the left crack tip.

Figure 3: Normalized moment intensity factors against location angle \( \phi \).

5 Numerical results

In this section, we consider the interaction problems as shown in Figs.3 and 4. The bending heat flux is assumed flowing along the \( y \)-axis direction \( (\delta=90^\circ) \), crack length is \( 2a \), the distance between the crack and ellipse centers is \( 2.3a \), and Poisson's ratio is 0.3. Moment intensity factors are calculated from eqns (36) and (37), and they are all normalized by a twisting moment intensity factor at right tip of a crack with length \( 2a \) under a perpendicular bending heat flux, whose value can be given by
\[ k_s = \frac{D(1 - \nu^2)q\alpha}{4k\sqrt{a}} \] (38)
The normalized moment intensity factors at tips A and B against the location angle \( \phi \) of the crack are shown in Fig.3 with the orientation angle of the crack \( \phi=\varphi \) and \( b=0.6a \). When the ellipse becomes a crack, variation of normalized moment intensity factors at tips C and D against the orientation angle \( \phi \) is shown in Fig.4.

When the crack locates far away the ellipse, the bending moment intensity factors \( (k_b) \) at both crack tips are zero. When it locates in the vicinity of the
ellipse, as shown in Figs. 3 and 4, interaction effect makes the bending moment intensity factor no longer to be zero. Moreover, it makes the twisting moment intensity factors of the two closer tips decrease, however, that for the two other tips increase. In the special case when the crack locates on the x-axis, the bending moment intensity factors are always zero. It means that the interaction for the perpendicular heat flux has no effect on the bending moment intensity factor.

6 Conclusion

This work performed a formulation for the interaction problem between an adiabatic elliptic hole and an arbitrarily located and orientated adiabatic crack in an infinite thin plate under uniform bending heat flux. A solution for the hole under uniform bending heat flux and a Green's function of bending heat source couple are derived. Two systems of singular integral equations are developed with the aid of the obtained fundamental solutions and the principle of superposition.

Numerical results considering the moment intensity factors were obtained. It showed that the interaction effect makes the bending moment intensity factor no longer to be zero, and the twisting moment intensity factors of two closer tips decrease, however, that for other tips increase. This phenomenon is quite different from that in problems under mechanical loadings.

References


