A multi-variable non-singular BEM in elasticity

E. Schnack\(^1\) & H.B. Chen\(^2\)

\(^1\)Institute of Solid Mechanics, Karlsruhe University, Germany
\(^2\)Department of Modern Mechanics, University of Science and Technology of China, P.R. China

Abstract

A new BEM algorithm is proposed in the paper. This algorithm is based on the coincident collocation of the non-singular boundary integral equations (BIEs) of displacement and its derivatives, and the basic variables of the new BEM system equation are the boundary nodal displacements, tractions and displacement derivatives. Due to the non-singular character of the BIEs, all integrals in the algorithm can be evaluated by the Gaussian quadrature. Furthermore, as the displacement derivative can be considered independent to the displacement in the formulations, the conforming element can be used for this algorithm. Once the new BEM system equation is solved, the boundary stress can be evaluated immediately from the obtained displacement derivatives. Three 2D numerical examples show the validity of the algorithm.

1 Introduction

Many important improvements in the BEM research in the past decade are concerned with the hyper-singular integral equations of the secondary fields (e.g. potential derivative, stress, etc.). For instance, the traction BIE plays a critical role in the fracture BEM analysis, as there the displacement BIE degenerates on the overlapping surface [1]. The hyper-singular BIE plays a crucial part in the analysis of exterior acoustic or elastic wave problems, where it is combined linearly with the conventional BIE to prevent the so-called fictitious eigenfrequencies [2]. For the stress recovery in the...
elastic BEM, though the stress BIE on smooth boundary has been found for a long time, its direct use for this purpose is just in the recent years [3]. Muci-Küchler and Rudolphi [4] use the regularized tangent derivative and the general displacement BIEs to form a Hermite interpolation scheme and the displacement tangent derivatives are obtained simultaneously with the unknown boundary displacements and tractions, therefore, an improved boundary stresses are recovered. It is found recently that the hyper-singular BIE for displacement derivative can be used directly and effectively for the boundary stress recovery, even in the commonly used $C^0$ elements [5].

For the use of hyper-singular BIEs, one important issue is the singularity treatment of the hyper and strongly singular integrals involved. The direct calculation method, see e.g. [6], is effective and has been used widely in the fracture BEM analysis, however, this method shows tedious, especially for the general structural problems. Therefore, many recent research are concerned with the regularized hyper-singular BIEs [7, 5]. Through the regularized formulations, the singular integrals are transferred to weakly or non-singular ones, and therefore, only the general Gaussian quadrature scheme is required for the integrations. Another important issue is what kind of element can be used, i.e. what kind of element meets the continuity requirement of the formulations. Theoretically, the displacement field is requested to be of a Hölder first derivative continuity around the source point, and therefore a $C^n$ ($n \geq 1$) element or a discontinuous element is expected for the formulations. Nevertheless, this stringent requirement has been shown to be able to be relaxed to a piecewise $C^1$ continuous [8, 2], thus the general $C^0$ element can be used for this formulation.

In the present paper, the regularized (or non-singular) displacement derivative BIE is used to collocate simultaneously with the fully regularized displacement BIE to form a new BEM system equation. In the new BEM system, the basic variables are the nodal displacements, tractions and displacement derivatives, where the displacement derivatives are considered independent to the displacements. Due to this variable consideration and the non-singular character of the formulations, the isoparametric conforming element and the Gaussian quadrature scheme can be used for this new algorithm. Three 2D examples show the validity of this algorithm.

2 Non-singular displacement BIE

In this paper, the formulations are considered with no body forces for simplicity. The displacement BIE for boundary point $p$ can be written as

$$c_{ij}(p)u_j(p) = \int_{\Gamma} u_{ij}^*(p, x)p_j(x)d\Gamma(x) - \int_{\Gamma} p_{ij}^*(p, x)u_j(x)d\Gamma(x). \quad (1)$$

where $c_{ij}(p)$ is the well-known displacement free term coefficient, which depends on the structural boundary shape at point $p$; $u_j(x)$ and $p_j(x)$ are
boundary displacements and tractions, respectively, at field point \( x \); \( \Gamma \) is the boundary of the considered domain \( \Omega \) in analysis; \( u_{ij}^*(p, x) \) and \( p_{ij}^*(p, x) \) are Kelvin’s fundamental solutions and the second integral on the right hand side of eqn (1) exist in the Cauchy principal value (CPV) sense.

Now assuming the considered domain subject to a rigid movement which value equals to \( u_j(p) \), thus

\[
p_j(x) = 0 \\
u_j(x) = u_j(p) \quad \{ \quad x \in \Gamma.
\]

From eqn (1), we have

\[
c_{ij}(p)u_j(p) = - \int_{\Gamma} p_{ij}^*(p, x)u_j(p) \, d\Gamma(x) + \lambda u_i(p).
\]

where

\[
\lambda = \begin{cases} 
0, & \text{for a finite problem;} \\
1, & \text{for an infinite problem.}
\end{cases}
\]

Furthermore, consider a uniform \( u_{i,k}(p) \) state, i.e. \( u_{i,k}(x) \equiv u_{i,k}(p), \ x \in \Omega \cup \Gamma \), and set the static point of the state at \( p \), we have

\[
p_j(x) = E_{jlmn} u_{m,n}(p) n_l(x) \\
u_j(x) = u_{j,l}(p) r_l(p, x) \quad \{ \quad x \in \Gamma.
\]

where \( E_{jlmn} \) is the elastic relationship between stress and displacement derivative. Therefore, in this state, from eqn (1) we have

\[
0 = \int_{\Gamma} u_{ij}^*(p, x) E_{jlmn} u_{m,n}(p) n_l(x) \, d\Gamma(x) - \int_{\Gamma} p_{ij}^*(p, x) u_{j,l}(p) r_l(p, x) \, d\Gamma(x).
\]

Subtracting eqns (3) and (6) from eqn (1), we get the fully non-singular boundary displacement BIE as

\[
\int_{\Gamma} p_{ij}^*(p, x)[u_j(x) - u_j(p) - u_{j,l}(p) r_l(p, x)] \, d\Gamma(x) + \lambda u_i(p) \\
= \int_{\Gamma} u_{ij}^*(p, x)[p_j(x) - E_{jlmn} u_{m,n}(p) n_l(x)] \, d\Gamma(x).
\]

3 Non-singular displacement derivative BIE

The boundary displacement derivative BIE can be expressed as [9]

\[
c'_{ik,jl}(p)u_{j,l}(p) = \int_{\Gamma} u_{ij,k}^*(p, x) p_j(x) \, d\Gamma(x) \\
- \int_{\Gamma} p_{ij,k}^*(p, x)[u_j(x) - u_j(p)] \, d\Gamma(x).
\]
if the boundary displacement derivative $u_{i,k}(x)$ satisfies a Hölder continuous condition at $p$. $c'_{ikjl}(p)$ is the free term coefficient of boundary displacement derivative, which is dependent only on the structural boundary shape at point $p$ too, and the two integrals in eqn (8) exist in the CPV sense. $u^*_{ij,k}(p, x)$ and $p^*_{ij,k}(p, x)$ are the differential forms of $u^*_{ij}(p, x)$ and $p^*_{ij}(p, x)$, respectively, with respect to the source point $p$ along coordinate $x_k$.

Now consider the uniform $u_{i,k}(p)$ state and set its static point at $p$, we get eqn (5) again, from eqn (8) we have

\[ c'_{ikjl} u_{j,l}(p) = \int \Gamma u^*_{ij,k}(\xi, x) E_{jlmn} u_{m,n}(p) n_l(x) d\Gamma(x) \]
\[ - \int \Gamma p^*_{ij,k}(\xi, x) u_{j,l}(p) r_l(p, x) d\Gamma(x) + \lambda u_{i,k}(p). \] (9)

Subtract eqn (9) from eqn (8), we get the non-singular BIE for boundary displacement derivative as

\[ \int \Gamma p^*_{ij,k}(p, x)[u_j(x) - u_j(p) - u_{j,l}(p) r_l(p, x)] d\Gamma(x) + \lambda u_{i,k}(p) \]
\[ = \int \Gamma u^*_{ij,k}(p, x)[p_j(x) - E_{jlmn} u_{m,n}(p) n_l(x)] d\Gamma(x). \] (10)

The non-singular formulations (7) and (10) can also be obtained as an extreme case from their respective regularized formulations for interior points [5], or from some special identities of the fundamental solutions [10, 11].

4 Numerical treatments

The new BEM algorithm proposed in the paper is based on the coincident collocation of eqns (7) and (10), and therefore, forms a new BEM equation system for boundary nodal displacements, tractions and displacement derivatives. For each node in a general structural problem, there are two nodal unknowns of displacements and/or tractions, and four unknowns of displacement derivatives. Meanwhile, eqns (7) and (10) offer two and four equations, respectively, for each node. Like the general BEM solution procedure, collocating eqns (7) and (10) at all discretized nodes and composing an equation system, we can get the boundary unknowns of displacements, tractions and displacement derivatives for all nodes at the same time by solving the new equation system.

An important issue for this algorithm is what kind of element can or should be used. The non-singular displacement BIE (7) has never been applied numerically, however, as it uses the displacement derivative $u_{i,k}(p)$ to regularize the singularity, the continuity requirement for this formulation is the same with the hyper-singular BIE (10). A previous widely held notion[12], that the standard conforming isoparametric boundary element may not be used in the solution of hyper-singular integral equations, mainly
comes from two aspects. The first one is, theoretically, the hyper-singular integral equations exist only when the primary field is $C^{1,\alpha}$ continuous, i.e. the Hölder derivative continuous, and therefore a $C^n (n \geq 1)$ interpolation is expected. The second one is, numerically, some algorithms based on the hyper-singular integral equations involve the numerical derivation of the primary field variable (displacement or potential), see e.g. [13, 7], and therefore, there will be inconsistent derivative values at the element joint nodes calculated from different joint elements, if the conforming element is used. However, directly applied or by some simple treatments [13, 7], the conforming element has been shown to be valid for the regularized hyper-singular integral equations and this was explained as certain categories of relaxation [8, 2].

The algorithm proposed in the paper does not involve any numerical derivation for boundary displacement, therefore, the displacement derivative can be regarded as an independent value from the displacement in the regularized formulations and both of them can be interpolated independently by the $C^0$ element. Therefore, numerically, the conforming element is rigorously applicable for the algorithm in the paper. It should be noted that, the traction constitutive equation at node $p$, i.e.

$$p_j(p) = E_{jimn}u_{m,n}(p)n_l(p).$$

should be enforced in the numerical implementation of eqns (7) and (10). This enforcement can avoid the inconsistency of the numerical outward normals at and near the source points.

5 Numerical tests

In this section, three 2D problems are analyzed to test the algorithm proposed in the paper. The material parameters for all the examples are $E = 2 \times 10^5 MPa$ and $\nu = 0.25$. In all examples, the isoparametric quadratic element and ten point Gaussian quadrature for each element are used. The examples are all analyzed in Cartesian coordinates and under double precisions for the variables.

Example 1: A square plate subject to uniform shear stress

A square plate with a side-length 12 cm subject to shear stress $\tau = 1 MPa$ on the whole boundary as shown in Figure 1. The stresses at any point of the plate are $\sigma_{11} = \sigma_{22} = 0$ and $\sigma_{21} = \sigma_{12} = 1 MPa$ under the Cartesian coordinates. If the plate is confined as shown in Figure 1, the displacement derivative solution of the whole plate is $u_{1,1} = u_{2,2} = u_{2,1} = 0$, $u_{1,2} = 0.125 \times 10^{-4}$. The plate is discretized into four quadratic elements, each side one element. Numerical results of the nodal displacement derivatives are in good accuracy, among them those at the corner nodes are of relative bigger errors and are listed in Table 1.
Figure 1: A square plate subject to uniform shear stress

Table 1. The displacement derivatives at the corner nodes of the square plate

<table>
<thead>
<tr>
<th>Node</th>
<th>$u_{1,1}$</th>
<th>$u_{2,2}$</th>
<th>$u_{2,1}$</th>
<th>$u_{1,2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.476456D-21</td>
<td>0.300432D-19</td>
<td>0.359989D-20</td>
<td>0.125000D-04</td>
</tr>
<tr>
<td>B</td>
<td>-0.728581D-20</td>
<td>0.180722D-19</td>
<td>0.201253D-19</td>
<td>0.125000D-04</td>
</tr>
<tr>
<td>C</td>
<td>-0.380040D-19</td>
<td>-0.481915D-19</td>
<td>-0.320020D-19</td>
<td>0.125000D-04</td>
</tr>
<tr>
<td>D</td>
<td>0.310888D-19</td>
<td>-0.181583D-19</td>
<td>0.827181D-19</td>
<td>0.125000D-04</td>
</tr>
<tr>
<td>Exact</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.125000D-04</td>
</tr>
</tbody>
</table>

Example 2: A hollow cylinder subject to internal pressure

The second example is a thick hollow cylinder with its inner and outer radii $a = 10\, cm$ and $b = 25\, cm$ subject to internal pressure $p = 100\, MPa$, as shown in Figure 2. By use of symmetry, a quarter of the cylinder is analyzed and three discretization schemes are considered as shown in Figure 3.

Figure 2: A thick cylinder with internal pressure,
Table 2. Boundary nodal stresses for internal press cylinder (MPa)

<table>
<thead>
<tr>
<th>Boundary Node</th>
<th>Stress Component</th>
<th>Exact Solution</th>
<th>Numerical results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \sigma_{11} )</td>
<td>19.048</td>
<td>18.931</td>
</tr>
<tr>
<td>A</td>
<td>( \sigma_{22} )</td>
<td>19.048</td>
<td>18.931</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{12} )</td>
<td>-119.048</td>
<td>-118.977</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{11} )</td>
<td>-100.000</td>
<td>-103.801</td>
</tr>
<tr>
<td>B</td>
<td>( \sigma_{22} )</td>
<td>138.095</td>
<td>132.469</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{12} )</td>
<td>0.000</td>
<td>-0.767</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{11} )</td>
<td>-19.825</td>
<td>-20.020</td>
</tr>
<tr>
<td>C</td>
<td>( \sigma_{22} )</td>
<td>57.920</td>
<td>57.897</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{12} )</td>
<td>0.000</td>
<td>-1.731</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{11} )</td>
<td>0.000</td>
<td>-2.903</td>
</tr>
<tr>
<td>D</td>
<td>( \sigma_{22} )</td>
<td>38.095</td>
<td>37.416</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{12} )</td>
<td>0.000</td>
<td>0.025</td>
</tr>
<tr>
<td>E</td>
<td>( \sigma_{11} )</td>
<td>19.048</td>
<td>18.150</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{22} )</td>
<td>19.048</td>
<td>18.150</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{12} )</td>
<td>-19.048</td>
<td>-19.023</td>
</tr>
</tbody>
</table>

Table 2 shows the nodal stresses at the five nodes, shown in Figure 3, and compared with the exact solution. It can be seen that, globally, the nodal stresses are in good accuracy and refined results can be obtained as the mesh is subdivided.

Example 3: A uniform tension infinite plate with a hole

The third example is an infinite plate with a circular hole of radius \( a = 10cm \) subject to a uniform horizontal tension \( \sigma = 1MPa \) at infinity, as shown in
Figure 4. This problem can be analyzed with the discretization only on the boundary line of the circular hole. To do this, the problem can be considered as a sum of the two cases shown in Figure 5 and, therefore, only Case II in Figure 5 needs to be analyzed by BEM. Again here, three discretization schemes, as shown in Figure 6, are considered to compare the results in different meshes.

![Diagram](image1)

Figure 4: An uniform tension infinite plate with a circular hole

![Diagram](image2)

Figure 5: Equivalent problem for the uniform tension infinite plate

![Diagram](image3)

Figure 6: Discretization schemes for the circular hole in the infinite plate
Table 3 gives the stresses of node A, B and C, shown Figure 6, and compared with the exact solutions. It can be seen that the numerical results are in good coincidence with the exact solutions and converge to the exact one as the mesh is subdivided. The numerical results in this example are in good accuracy in a coarser discretization compared with example 2, as here the boundary nodes are all structural smooth points.

<table>
<thead>
<tr>
<th>Boundary Node</th>
<th>Stress Component</th>
<th>Exact Solution</th>
<th>Numerical results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Mesh I</td>
</tr>
<tr>
<td>A</td>
<td>(\sigma_{11})</td>
<td>3.0000</td>
<td>2.9972</td>
</tr>
<tr>
<td></td>
<td>(\sigma_{22})</td>
<td>0.0000</td>
<td>-0.0028</td>
</tr>
<tr>
<td></td>
<td>(\sigma_{12})</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>B</td>
<td>(\sigma_{11})</td>
<td>0.5000</td>
<td>0.4900</td>
</tr>
<tr>
<td></td>
<td>(\sigma_{22})</td>
<td>0.5000</td>
<td>0.5029</td>
</tr>
<tr>
<td></td>
<td>(\sigma_{12})</td>
<td>-0.5000</td>
<td>-0.4986</td>
</tr>
<tr>
<td>C</td>
<td>(\sigma_{11})</td>
<td>0.0000</td>
<td>0.0220</td>
</tr>
<tr>
<td></td>
<td>(\sigma_{22})</td>
<td>-1.0000</td>
<td>-0.9786</td>
</tr>
<tr>
<td></td>
<td>(\sigma_{12})</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

6 Conclusions

A new BEM algorithm is proposed in the paper, in which the basic boundary variables are the nodal displacements, tractions and displacement derivatives. This algorithm is based on the non-singular BIEs of displacement and its derivatives. Due to the fully non-singular character of the formulations, not any singularity treatment, even the logarithmic singularity treatment in the traditional 2D BEM, is needed in the numerical implementation, and therefore, all the integrations are proceeded by the Gaussian quadrature. And as the displacement derivative can be considered independent to the displacement in the formulations, the commonly used conforming element is rigorously applicable for this algorithm.

Once the new system equation is solved, the boundary displacement derivatives, together with the boundary unknown displacements and tractions, are obtained and therefore the boundary stresses can be recovered directly. Numerical tests show that this algorithm reaches high accuracy for boundary stresses. A drawback of this algorithm is that the unknown variable of the new equation system is three times of that in the traditional BEM, thus more computation time is needed to calculate the equation coefficients and to solve the equation system.
Acknowledgments

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References


