Analytic solutions of the diffusion-deposition equation for fluids heavier than atmospheric air

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Abstract

A steady-state bi-dimensional turbulent diffusion equation was studied to find the concentration distribution of a pollutant near the ground. We have considered the air pollutant emitted from an elevated point source in the lower atmosphere in adiabatic conditions. The wind velocity and diffusion coefficient are given by power laws. We have found analytical solutions using or the Lie Group Analysis or the Method of Separation of Variables. The classical diffusion equation has been modified introducing the falling term with non-zero deposition velocity.

Analytical solutions are essential to test numerical models for the great difficulty in validating with experiments.
Keywords: atmospheric pollution, diffusion equation, exact solutions.

1 Introduction

The classical form of the mean steady diffusion equation is valid for elementary particles of the fluid or when the foreign particles are of the same density as the fluid. If the density and dimensions are high enough to have terminal velocities v_s not negligible, the distribution of the particles will be affected in various ways [1–4].

A simple approximation is to consider that the particle sinks at a rate v_s and the ground acts as a permeable surface and retains all material passing through it.

Using a very simple model it is possible to examine various cases. It is possible to have exact solutions of the mean steady diffusion when the turbulent diffusivity k_z and the terminal velocity v_d depend somehow on the height [5–7].

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2 Mathematical model

The sedimentation of the material may be allowed by introducing a convection term in the mean steady equation that becomes [8, 9]:

$$u(z)\frac{\partial c}{\partial x} = \frac{\partial}{\partial z}\left(k_z(z)\frac{\partial c}{\partial z} + v_s(z)c\right) \tag{1}$$

where $v_s(z)$ is the deposition velocity.

We assume that mean wind velocity $u(z)$, the eddy diffusivity in z-direction $k_z(z)$ and the deposition velocity $v_s(z)$ are:

$$u(z) = u_0 z^{\alpha} \tag{2}$$

$$k_z(z) = k_0 z^{n} \tag{3}$$

$$v_s(z) = v_0 z^{q} \tag{4}$$

3 Group analysis of the equation

Group analysis of the (1) is performed through the one-parameter Lie group of transformations:

$$\begin{cases} x^* = x + \epsilon X(x, z, c) + O(\epsilon^2) \\ z^* = z + \epsilon Z(x, z, c) + O(\epsilon^2) \\ c^* = c + \epsilon C(x, z, c) + O(\epsilon^2) \end{cases} \tag{5}$$

where X, Z, C are the infinitesimal generators of the transformations [10–12].

Equation (1) is invariant respect to the group (5) of transformations if c^* is the solution of eq. (1) in the star variables. In this case, the number of independent variables can be decreased.

A considerable difficulty lies in the amount of the auxiliary calculations involved. We performed the calculations of the generators of the transformations group on a P.C. using the *MATHEMATICA* package.

Since eq. (1) is linear, the infinitesimal generators of the group of invariance are of the form:

$$\begin{cases} X = X(x) \\ Z = Z(x, z) \\ C = A(x, z)c + B(x, z) \end{cases} \tag{6}$$

The function $B(x, z)$ must satisfy eq. (1) and, without compromising with the generality, can be assumed equal to zero.

If we normalize the parameters $u_0/k_0 \rightarrow v_0$ and the variable $k_0 x/v_0 \rightarrow x$, we have that X, Z and A must satisfy the following equations:

$$nz^{-1}Z - z^{-1}\alpha Z + X' - 2Z_z = 0 \tag{7}$$

$$qv_0z^{-2-n+q}Z + nqv_0z^{-2-n+q}Z - q^2v_0z^{-2-n+q}Z - nz^{-1}A_z$$
$$- v_0z^{-n+q}A_z - 2qv_0z^{-1-n+q}Z_z - A_{zz} + z^{-n+\alpha}A_x = 0 \quad (8)$$
$$nz^{-2}Z + nv_0z^{-1-n+q}Z - qv_0z^{-1-n+q}Z - 2A_z - nz^{-1}Z_z$$
$$- v_0z^{-n+q}Z_z + Z_{zz} - z^{-n+\alpha}Z_x = 0 \quad (9)$$

We show now some results.

4 Similarity solutions

Let us look at some similarity solutions.

4.1 α, n and q arbitrary ($n - \alpha \neq 2$)

In this case it possible to obtain from the eq.s (7–9) the generators of group of similarity:

$$\begin{cases} X = a_0 \\ Z = 0 \\ C = c_1 c \end{cases} \quad (10)$$

where a_0, and c_1 are arbitrary constants.

The characteristic equations are:

$$\frac{\mathrm{d}x}{x_1} = \frac{\mathrm{d}z}{0} = \frac{\mathrm{d}c}{c_1 c}$$

The invariants are z and $c\mathrm{e}^{-\frac{c_1}{x_1}x}$. If we assume $\frac{c_1}{x_1} = -\lambda^2$, the similarity solution, corresponding to the separation of variables, is

$$c = \mathrm{e}^{-\lambda^2 x} Z(z)$$

where $Z(z)$ is solution of the following ordinary differential equation:

$$(qv_0z^{1-n+q} + z^{2-n+\alpha}\lambda^2)Z(z) + z(n + v_0z^{1-n+q})Z'(z) + z^2Z''(z) = 0$$

If $n = 2q - \alpha$, the solution is [13]

$$Z(z) = \mathrm{e}^{\frac{z^{1-q+\alpha}(v_0+\sqrt{v_0^2-4\lambda^2})}{-2+2q-2\alpha}}$$
$$\times \left[h_1 \Psi\left(\frac{2 - 3q + 2\alpha - \frac{qv_0}{\sqrt{v_0^2-4\lambda^2}}}{2(1-q+\alpha)}, \frac{2-3q+2\alpha}{1-q+\alpha}; \frac{z^{1-q+\alpha}\sqrt{v_0^2-4\lambda^2}}{1-q+\alpha} \right) \right.$$
$$\left. + h_2 L\left(\frac{2 - 3q + 2\alpha - \frac{qv_0}{\sqrt{v_0^2-4\lambda^2}}}{-2(1-q+\alpha)}, \frac{1-2q+\alpha}{1-q+\alpha}; \frac{z^{1-q+\alpha}\sqrt{v_0^2-4\lambda^2}}{1-q+\alpha} \right) \right]$$

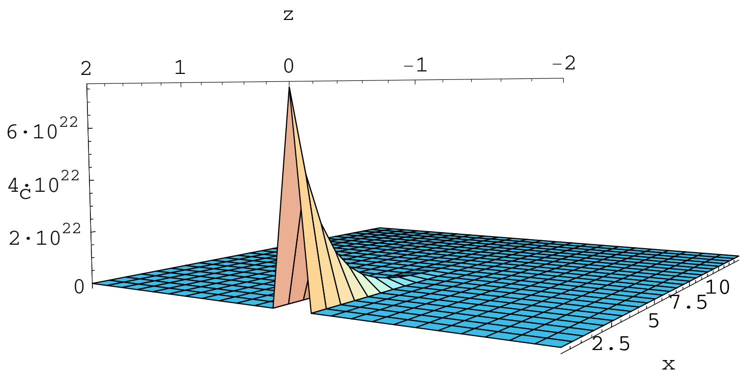

Figure 1: The $c(x, z)$, $v_0 = 20$, $q = 0.5$, $\lambda = 1$, $\alpha = 1.5$, $h_1 = 1$, $h_2 = 0$.

where h_1 and h_2 are arbitrary constants, $\Psi(-, -; \cdot)$ is the confluent hypergeometric function and $L(-, -; \cdot)$ is the generalized Laguerre polynomial. The concentration is

$$c = e^{-\lambda^2 x} e^{-\frac{(v_0+\sqrt{v_0^2-4\lambda^2})}{1-q+\alpha} \frac{z^{1-q+\alpha}}{2}}$$

$$\times \left[h_1 \Psi\left(\frac{2-3q+2\alpha - \frac{qv_0}{\sqrt{v_0^2-4\lambda^2}}}{2(1-q+\alpha)}, \frac{2-3q+2\alpha}{1-q+\alpha}; \frac{z^{1-q+\alpha}\sqrt{v_0^2-4\lambda^2}}{1-q+\alpha} \right) \right.$$

$$\left. + h_2 L\left(\frac{2-3q+2\alpha - \frac{qv_0}{\sqrt{v_0^2-4\lambda^2}}}{-2(1-q+\alpha)}, \frac{1-2q+\alpha}{1-q+\alpha}; \frac{z^{1-q+\alpha}\sqrt{v_0^2-4\lambda^2}}{1-q+\alpha} \right) \right]$$

4.2 $n = 1$, $q = 0$ and α arbitrary

We observe that in this case $k_0 = ku_*$ where k is the Von Karman constant and u_* is the friction velocity.

The generators of group of similarity are:

$$\begin{cases} X = a_0 + a_1 x + \dfrac{a_2}{2} x^2 \\ Z = \dfrac{a_1 + a_2 x}{1+\alpha} z + c_0 z^{\dfrac{1-\alpha}{2}} \\ C = (b_0 + b_1 z^{1+\alpha} + b_1 x (1+\alpha)(1+v_0+\alpha))c \end{cases} \tag{11}$$

where a_0, a_1, a_2, b_0, b_1 and c_0 are constants satisfying the conditions

$$c_0(1 + 2v_0 + \alpha) = 0, \quad a_2 + 2b_1(1+\alpha)^2 = 0$$

Now we consider the following subcases.

4.2.1 $c_0 = 0, a_2 = -2b_1(1+\alpha)^2$ and $\alpha \neq -1$

The characteristic equations are:

$$\frac{dx}{x} = (1+\alpha)\frac{dz}{z} = \frac{dc}{c}$$

The invariants are $\frac{c}{x}$ and $\xi = zx^{-\frac{1}{1+\alpha}}$; the concentration is

$$c = xf(\xi)$$

where $f(\xi)$ is solution of the following ordinary differential equation:

$$\xi^{\alpha} f(\xi) - \left(1 + v_0 + \frac{\xi^{1+\alpha}}{1+\alpha}\right) f'(\xi) - \xi f''(\xi) = 0$$

In this case we have:

$$f(\xi) = e^{-\frac{z^{1+\alpha}}{x(1+\alpha)^2}} \left[h_1 \Psi\left(\frac{2+v_0+2\alpha}{1+\alpha}, 1 + \frac{v_0}{1+\alpha}; \frac{z^{1+\alpha}}{x(1+\alpha)^2}\right) + h_2 L\left(-\frac{2+v_0+2\alpha}{1+\alpha}, \frac{v_0}{1+\alpha}; \frac{z^{1+\alpha}}{x(1+\alpha)^2}\right)\right]$$

where h_1 and h_2 are arbitrary constants. The concentration is

$$c = xe^{-\frac{z^{1+\alpha}}{x(1+\alpha)^2}} \left[h_1 \Psi\left(\frac{2+v_0+2\alpha}{1+\alpha}, 1 + \frac{v_0}{1+\alpha}; \frac{z^{1+\alpha}}{x(1+\alpha)^2}\right) + h_2 L\left(-\frac{2+v_0+2\alpha}{1+\alpha}, \frac{v_0}{1+\alpha}; \frac{z^{1+\alpha}}{x(1+\alpha)^2}\right)\right]$$

4.2.2 $(1 + 2v_0 + \alpha) = 0, a_0 = -1, b_1 = 0$ and $\alpha \neq -1$

The characteristic equations are:

$$\frac{dx}{-1} = \frac{dz}{z^{\frac{1-\alpha}{2}}} = \frac{dc}{c}$$

The invariants are ce^{x} and $\xi = x + \frac{2}{1+\alpha}z^{\frac{\alpha+1}{2}}$; the concentration is

$$c = e^{-x} f(\xi)$$

where $f(\xi)$ is solution of the following ordinary differential equation:

$$f(\xi) - f'(\xi) + f''(\xi) = 0$$

In this case the concentration is:

$$f(\xi) = e^{-\frac{x}{2} + \frac{z^{\frac{1+\alpha}{2}}}{1+\alpha}} \left[h_1 \cos\sqrt{3}\left(\frac{x}{2} + \frac{z^{\frac{1+\alpha}{2}}}{1+\alpha}\right) + h_2 \sin\sqrt{3}\left(\frac{x}{2} + \frac{z^{\frac{1+\alpha}{2}}}{1+\alpha}\right)\right]$$

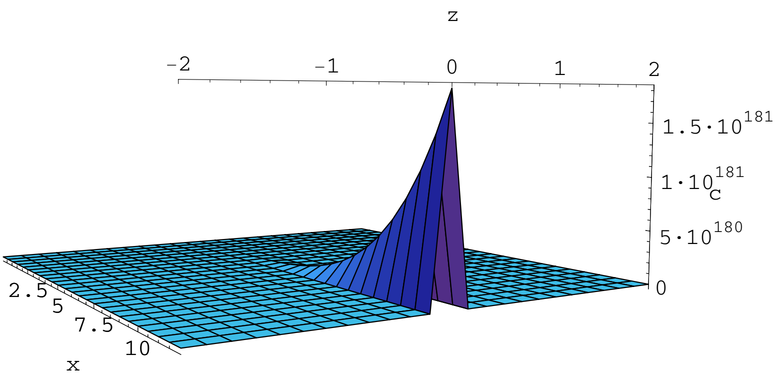

Figure 2: The $c(x, z)$, $v_0 = 11, \alpha = 1, h_1 = 1, h_2 = 0$.

4.3 $n = 2, q = 1$ and $\alpha = 0$

The generators of group of similarity are:

$$\begin{cases} X = a_0 + a_1 x + \dfrac{a_2}{2}x^2 \\ Z = b_0 z + \dfrac{1}{2}(a_1 + a_2 x) z \log z \\ C = \Bigg(c_2 - \dfrac{1}{8}x(2a_1(v_0 - 1)^2 + a_2(2 + (v_0 - 1)^2 x)) \\ \qquad - \dfrac{1}{8}\log z(2(v_0 + 1)(a_1 + a_2 x) + a_2 \log z)\Bigg)c \end{cases} \tag{12}$$

where a_0, a_1, a_2, b_0, and c_2 are arbitrary constants.
If we put: $a_0 = 0, a_1 = 1, a_2 = 0, b_0 = 0, c_2 = 0$, we have

$$\begin{cases} X = x \\ Z = \dfrac{1}{2}z \log z \\ C = -\dfrac{1}{4}((v_0 - 1)^2 x + (1 + v_0)\log z)c \end{cases} \tag{13}$$

The invariants are $ce^{\frac{1}{4}(v_0-1)^2 x} z^{\frac{1}{2}(1+v_0)}$ and $\xi = \frac{\log z}{\sqrt{x}}$; the concentration is

$$c = \mathrm{e}^{-\frac{1}{4}(v_0-1)^2 x} z^{-\frac{1}{2}(1+v_0)} f(\xi)$$

where $f(\xi)$ is solution of the following ordinary differential equation:

$$f''(\xi) + \frac{\xi}{2}f'(\xi) = 0$$

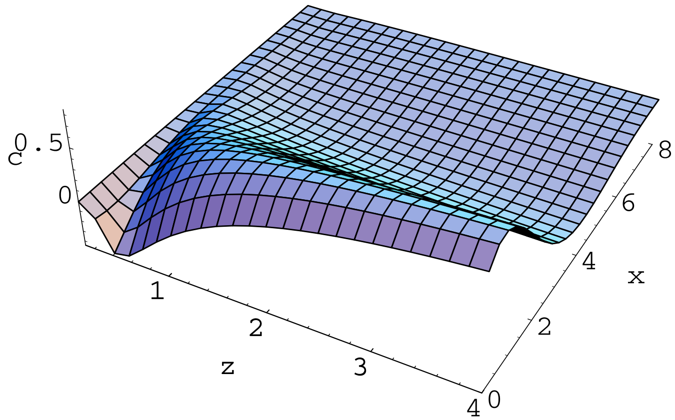

Figure 3: The $c(x, z)$, $\alpha = -3$, $h_1 = 1$, $h_2 = 1$.

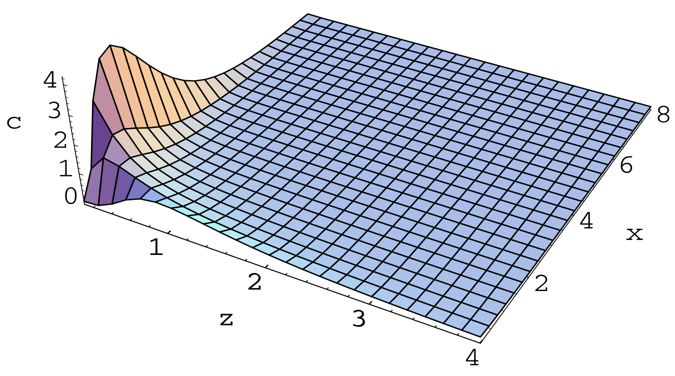

Figure 4: The $c(x, z)$, $v_0 = 3$, $h_1 = 1$, $h_2 = 1$.

the solution is

$$f(\xi) = h_1 + h_2 \operatorname{erf}\left(\frac{\xi}{2}\right)$$

The concentration is

$$c = e^{-\frac{1}{4}(v_0-1)^2 x} z^{-\frac{1}{2}(1+v_0)} \left(h_1 + h_2 \operatorname{erf}\left(\frac{1}{2}\frac{\log z}{\sqrt{x}}\right)\right)$$

5 Conclusion

We obtain analytical solutions, using Lie group analysis, for steady-state bi-dimensional turbulent diffusion equation with variable coefficients. The laws of wind speed, turbulent diffusion coefficients and terminal velocities are specified by power laws. The obtained solutions are more realistic respect to gaussian model in air pollution modeling. In future we intend, using our solutions, to solve the diffusion equation (1) for many boundary conditions.

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