Effect of a constant and uniform magnetic field on Rayleigh-Bénard instabilities in cylindrical cavities

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Abstract

The effect of a constant and uniform magnetic field on three dimensional steady flow is numerically simulated in cylindrical cavity of aspect ratio \( A = H/D \) (\( H \) is the height and \( D \) the diameter). The flow patterns at thresholds are generally identified with respect to their azimuthal variations in \( \exp(i m \theta) \). Without magnetic field, basic axisymmetric (\( m = 0 \)) and asymmetric (\( m = 1 \) and \( m = 2 \)) modes are obtained. For \( A = 0.5 \), the axisymmetric flow becomes unstable at a secondary bifurcation point. The use of a vertical magnetic field keeps the same convective modes, but they are not equally stabilized. The horizontal magnetic field breaks some symmetries of the flow. Therefore, the convective modes are changed and the secondary bifurcation disappears.

1 Introduction

The purpose of this paper is the better understanding of the stabilizing action of a constant magnetic field on convective instabilities in cylindrical cavities heated from below. Our interest in
such study is connected to crystal growth applications where convective instabilities are known to alter the quality of crystals. For the pure thermal case in a cylindrical cavity, the analysis of the convective instabilities has been explored by Charlson & Sani\textsuperscript{1}, Buell & Catton\textsuperscript{2}, and Wanschura et al.\textsuperscript{3}. On the other hand, the effect of a constant magnetic field on electrically conducting liquid-metal flows in heated cavities has been mainly studied for lateral heating corresponding to horizontal Bridgman crystal growth configurations (BenHadid et al.\textsuperscript{4}, BenHadid & Henry\textsuperscript{5}). The influence of a vertical magnetic field on a convection arising in a fluid layer heated from below has been first investigated by Chandrasekhar\textsuperscript{6} for an infinite layer, then more recently by Mössner\textsuperscript{7} for a parallelepipedic cavity, but not up to now for a cylindrical cavity. We present in this paper the influence of the magnetic field on the primary and secondary instabilities in cylindrical cavities heated from below. Two orientations of the magnetic field are considered, either vertical or horizontal.

2 Mathematical model

We consider an incompressible fluid confined in a vertical cylindrical cavity (figure 1). The two ends of the cylinder are supposed to be isothermal with the lower one held at temperature $T_h$, which is greater than the temperature $T_c$ of the upper one. The sidewalls are considered to be adiabatic. The fluid is assumed to be Newtonian, electrically conducting and can be submitted to a uniform magnetic field $B_0$ which generates a damping Lorentz force through the creation of the induced electric current.

\textit{Figure 1: Cavity configuration.
The governing equations (Navier-Stokes, energy, Ohm’s law and continuity of electric current density) are made dimensionless using $D$, $D^2/\nu$, $U_{ref} = \nu Gr/D$, $B_0$ and $\sigma_e U_{ref} B_0$ as scales for length, time, velocity, magnetic flux density and electric current density respectively, and introducing the dimensionless temperature field as $\theta = (T-T_0)/(T_h-T_c)$ ($T_0$ is the mean temperature: $T_0 = (T_h+T_c)/2$). Finally, these equations, including the Lorentz force and with the Boussinesq approximation, can be written as:

$$\frac{\partial \mathbf{u}}{\partial t} = -Gr (\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \nabla^2 \mathbf{u} + \theta \hat{z} + Ha^2 \mathbf{J} \times \mathbf{e}_{B_0}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

$$\frac{\partial \theta}{\partial t} = -Gr \mathbf{u} \cdot \nabla \theta + \frac{1}{Pr} \nabla^2 \theta, \quad (3)$$

$$\mathbf{J} = E + \mathbf{u} \times \mathbf{e}_{B_0}, \quad (4)$$

$$\nabla \cdot \mathbf{J} = 0, \quad (5)$$

where $Gr$, $Pr$ and $Ha$ are the Grashof, the Prandtl and the Hartmann numbers defined respectively as $Gr = g\beta(T_h-T_c)D^4/H\nu^2$, $Pr = \nu/\kappa$ and $Ha = B_0 D(\sigma_e/\rho\nu)^{1/2}$. Combining (4) and (5), an equation for the electric potential $\Phi (E = -\nabla \Phi)$ can be obtained:

$$\nabla^2 \Phi = \nabla \cdot (\mathbf{u} \times \mathbf{e}_{B_0}). \quad (6)$$

The boundary conditions (no-slip, thermally insulated at the lateral wall, electrically insulated) can be written in their scaled form as:

\begin{align*}
\text{at } r &= 1/2, \quad u = v = w = \frac{\partial \theta}{\partial n} = \mathbf{J} \cdot \mathbf{n} = 0 \\
\text{at } z &= 0, \quad u = v = w = \theta - A/2 = \mathbf{J} \cdot \mathbf{n} = 0 \\
\text{at } z &= A, \quad u = v = w = \theta + A/2 = \mathbf{J} \cdot \mathbf{n} = 0
\end{align*} \quad (7-9)

Solving the above system for $\mathbf{u} = 0$, we get the temperature profile of the static solution $\theta(z) = (A/2 - z)$, which corresponds to the diffusive regime. The resolution of the governing equations is performed by an isoparametric spectral element method (Kardianakis et al.) for space discretization, and a finite difference
A continuation method based on a Newton solver is used to calculate the bifurcation diagrams giving the nonlinear evolution of the convection beyond its onset and the stability diagrams giving the evolution of the primary thresholds as a function of the aspect ratio or the Hartmann number. These primary thresholds are given through the critical value of the Rayleigh number $Ra$ ($Ra = GrPr$) and do not depend on the Prandtl number.

3 Results

3.1 Pure thermal case

The evolution of the primary thresholds, corresponding to the modes $m = 0, 1$ and 2, as a function of the aspect ratio $A$, is given in figure 2. We can notice that the convection sets in with an axisymmetric mode for $A < 0.55$ and with the $m = 1$ mode for $A > 0.55$. At $A = 0.55$, called double point, two patterns become unstable at the same value of $Ra_c$, and the linear analysis is ambiguous in predicting the fastest growing pattern. We also notice the crossing between $m = 0$ and $m = 2$ modes for $A = 0.63$.

![Figure 2: Influence of the aspect ratio on the primary thresholds for the pure thermal case.](image)
The bifurcation diagram given in figure 3 corresponds to $A = 0.5$ and $Pr = 1$. In this diagram the evolution of the vertical velocity $w$ at a fixed point is presented as a function of the Grashof number $Gr$. Above a critical value $Gr = 35854$, convection sets in with the axisymmetric mode ($m = 0$). The second branch originates at $Gr = 38928$ and corresponds to the $m = 2$ mode. Finally, the $m = 1$ branch appears at $Gr = 41783$. The most important result in this diagram is the secondary pitchfork bifurcation point for $Gr = 45708$. If the Grashof number exceeds this critical value, the axisymmetric solution ($m = 0$) becomes unstable and bifurcates to the stable asymmetric solution corresponding to a new mode ($m = 02$). This mode can be identified as a superposition of two modes: the mode $m = 0$, which is the basic mode, and the mode $m = 2$, the unstable eigenvector mode. The plot of the isolataches of the vertical velocity of the new solution shows that this mode consists of two counter-rotating rolls.

Figure 3: Bifurcation diagram for the pure thermal case ($A = 0.5$ and $Pr = 1$).
3.2 Vertical magnetic field

The use of a vertical magnetic field keeps the same symmetries as in the pure thermal case. The same modes $m = 0$, 1 and 2 are then obtained at the first primary thresholds. The variation of these primary thresholds as a function of $Ha$ is given in figure 4 for $A = 0.5$. The three modes are not identically stabilized; the mode $m = 2$ appears to be the less efficiently stabilized and becomes the most unstable for sufficiently large $Ha$ ($Ha > 24$ for $A = 0.5$). For $A = 0.5$, the same secondary bifurcation as in the pure thermal case is obtained. In figure 4, we give also the evolution of this secondary threshold $Rac_2$, as a function of the Hartmann number $Ha$ for the vertical magnetic field. We can easily observe that the difference $(Rac_2 - Rac_{(m=0)})$ decreases with $Ha$, so that we can conclude that the vertical magnetic field is unfavourable to the axisymmetric flow. We notice also that the primary bifurcations $Rac_{(m=0)}$ and $Rac_{(m=2)}$ intersect with the secondary bifurcation $Rac_2$ at $Ha=24$. It is a classical result where the secondary bifurcation appears or disappears at the intersection of the primary bifurcations (Dijkstra$^{10}$).

![Figure 4: Evolution of the primary thresholds ($Rac_{(m=0)}$, $Rac_{(m=1)}$ and $Rac_{(m=2)}$) and the secondary threshold ($Rac_2$) as a function of the Hartmann number (vertical magnetic field, $A = 0.5$).](image-url)
3.3 Horizontal magnetic field

The use of a horizontal magnetic field changes the symmetry properties of the configuration: three reflection symmetries are now effective, two with respect to the two vertical midplanes either parallel or orthogonal to the applied magnetic field (denoted respectively $P_\parallel$ and $P_\perp$ in figure 1), and one with respect to the horizontal midplane. The rotational degeneracy is then suppressed, but two favoured directions appear. As can be seen in figure 5, the three original modes ($m = 0, 1$ and 2) give with the horizontal magnetic field five distinct modes. For $A = 0.5$, the stabilization of the axisymmetric mode gives a mode with two counter-rotating rolls with axis parallel to $B_0$ (denoted $m = 02_\parallel$ as it can be seen as the superposition of the two modes $m = 0$ and $m = 2$). The stabilization of the asymmetric modes gives two modes for both of them: $m = 2$ and $m = 02_\perp$ for the mode $m = 2$ and $m = 1_\parallel$ and $m = 1_\perp$ for the mode $m = 1$. These modes are differently stabilized when $Ha$ is increased (figure 5), weakly if the axis of the rolls is parallel to $B_0$ and strongly if the axis of the rolls is perpendicular to $B_0$.

![Figure 5: Effect of the Hartmann number on the primary thresholds for the horizontal magnetic field (A = 0.5).](image)
Without magnetic field, for \( A = 0.5 \), at the secondary bifurcation the axisymmetric mode loses its stability and the structure of the new three-dimensional stable solution is identified as the \( m = 02 \) mode. When the horizontal magnetic field is applied, the structure of the emerging flow already corresponds to the mode \( m = 02_\parallel \). In terms of symmetry, the magnetic field breaks the symmetries which were broken by the secondary bifurcation in the pure thermal case. Thus, the secondary bifurcation disappears, and two disconnected branches are obtained, signature of an imperfect bifurcation (figure 6). The first branch of solutions corresponding to the \( m = 02_\parallel \) mode is stable. The second branch of solutions corresponds to the mode \( m = 02_\perp \) and is unstable. We note \( Ra_c' \) the critical value of \( Ra \) where the second branch appears and the number of solutions changes from one to three (saddle-node point). At \( Ra_c' \), the stability of the \( m = 02_\perp \) solutions changes. The first part of the branch which results from the originally axisymmetric solution is two time unstable, and the second part is one time unstable.

![Figure 6: Bifurcation diagram with horizontal magnetic field: imperfect secondary bifurcation. (\( Ha = 1, A = 0.5 \) and \( Pr = 1 \)).](image-url)
4 Discussion

The application of a magnetic field leads as expected to the stabilization of the cylindrically confined Rayleigh-Bénard situation. But as may be seen in figure 7 where the critical thresholds are given as a function of the aspect ratio at $Ha = 0$, and at $Ha = 20$ for both magnetic field orientations, the stabilization is much stronger with a vertical magnetic field.

This fact can be connected to the symmetry properties mentioned in the previous section. The vertical magnetic field can efficiently act on the different modes because the roll axis is in any case perpendicular to the applied field. On the contrary, the horizontal magnetic field privileges two orientations of the modes among which the one corresponding to rolls with axis parallel to the applied field which will be weakly stabilized and will appear as the critical one.

Figure 7: Comparison between the stabilization for vertical and horizontal magnetic fields.

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References


