The outflow boundary condition for mixed convection problems
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Abstract

The appropriate modelling of outflow boundary conditions in mixed convection flows with strong buoyancy effects, is a very challenging task. In this paper, “advective boundary conditions” are employed in the context of a recently proposed formulation based on primitive variables. The accuracy of the approach is demonstrated by the solution of a traditional benchmark problem: the two-dimensional Poiseuille-Benard flow in a horizontal channel heated from below.

1 Introduction

The biggest difficulty in the specification of outflow boundary conditions is brought about by the necessity of truncating the computational domain in order to make the problem tractable. When strong buoyancy effects are present, the task is so challenging that the Poiseuille-Benard channel flow problem (PBCF) is considered one of the most significant benchmarks to evaluate the performance of outflow boundary conditions in computational fluid dynamics [1-3]. In fact, the PBCF problem involves a time dependent flow in an open-ended horizontal channel, heated from below under conditions which result in a thermoconvective instability. To avoid solving the mass, momentum and energy conservation equations in a semi-infinite region, one must allow the transverse travelling waves to leave the domain through an artificial boundary placed in a strongly recirc-
Our research group has already solved the PBCF problem in the context of the streamfunction-vorticity formulation [4]. Here we illustrate a solution of the same problem in the context of a recently proposed primitive variable formulation [5-7]. At the outflow we employ advective conditions, both for velocity components and for temperature. We utilize a finite element Bubnov-Galerkin method for the space discretization, but the proposed algorithm is not tied to any particular discretization technique. The computational results presented show good agreement with the benchmark solution [2] and the streamfunction-vorticity solution [4].

2 Governing Equations and Solution Strategy

With reference to the Boussinesq approximation, the momentum and continuity equations which govern the laminar flow of a constant property fluid in a horizontal channel heated from below, can be written in dimensionless form as

\[
\frac{\partial \mathbf{v}}{\partial \vartheta} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{\text{Re}} \nabla^2 \mathbf{v} - \nabla p - \frac{1}{\text{Fr}} \frac{g}{t} \mathbf{g}
\]  

and

\[
\nabla \cdot \mathbf{v} = 0
\]

respectively. In the absence of volumetric heating and neglecting the effects of viscous dissipation, the energy equation can be written as

\[
\frac{\partial t}{\partial \vartheta} + \mathbf{v} \cdot \nabla t = \frac{1}{\text{Pe}} \nabla^2 t
\]  

The boundary conditions for the dimensionless velocity components \( u \), in the horizontal direction, and \( v \), in the vertical direction, are illustrated in Fig. 1 (a), while the boundary conditions for the dimensionless temperature are shown in Fig. 1 (b). The advective conditions at the artificial outflow boundary are written in terms of a constant phase speed, estimated as the average flow velocity. It must be pointed out that, in stationary problems, these advective conditions yield the “natural” zero normal derivative conditions.

The governing equations and the boundary conditions are nondimensionalized by using the channel height \( <H> \), the average flow
velocity \langle \overrightarrow{\mathbf{u}} \rangle$, the temperature difference \( \Delta t \) between the bottom and the top wall of the channel and a reference temperature coinciding with the temperature of the top wall. The Reynolds (Re), Froude (Fr) and Péclet (Pe) numbers are defined in the usual way, employing the reference quantities above defined [2,4].

The solution steps are derived in a continuous setting, where no reference is made to the particular space discretization used afterwards. A tentative pressure \( p^* \) is estimated first, at each new time step \( (n+1) \), by discretizing in time eqn (1) and assuming that the acceleration \( \frac{\partial \mathbf{v}}{\partial \vartheta} \) can be expressed as the sum of two contributions

\[
\hat{\mathbf{v}}_n - \mathbf{v}^{n} = \nabla \cdot \mathbf{v}^{n} = \frac{1}{2 \text{Re}} \nabla^2 (\hat{\mathbf{v}} + \mathbf{v}^{n}) - \frac{1}{\text{Fr}} t^n \frac{g}{g}
\]

and

\[
\frac{\mathbf{v}^{n+1} - \hat{\mathbf{v}}}{\Delta \vartheta} = -\nabla p^*
\]
In the above equations, \( \hat{v} \) is the pseudovelocity, i.e., the velocity that would prevail in the absence of the pressure field, and \((\Delta \vartheta)\)' is the time step utilized for estimating the pressure \( p^* \). It must be pointed out that in eqn (4) the advective term is dealt with explicitly, while the diffusion term is approximated by the Crank-Nicolson scheme.

Taking the divergence of eqn (5) and assuming

\[
\nabla \cdot v^{n+1} = 0
\]

as a means of enforcing continuity through the action of \( \nabla p^* \), we obtain

\[
\nabla^2 p^* = \frac{1}{(\Delta \vartheta)'} \nabla \cdot \hat{v}
\]

i.e., the Poisson equation for the estimated pressure.

The estimated pressure is used to solve the momentum equations by means of the Crank-Nicolson scheme

\[
\frac{v^* - v^n}{\Delta \vartheta} + \frac{1}{2} v^n \cdot \nabla (v^* + v^n) = \frac{1}{2 \text{Re}} \nabla^2 (v^* + v^n) - \nabla p^* - \frac{1}{\text{Fr}} t^n g
\]

where \( v^* \) is the velocity field that corresponds to \( p^* \) and the time step \( \Delta \vartheta \) can be different from \((\Delta \vartheta)\)' . Since \( v^* \), in general, does not respect continuity, we can find pressure corrections \( p' \) which project \( v^* \) onto the divergence-free space \( v^{n+1} \) by writing

\[
\frac{v^{n+1} - v^*}{\Delta \vartheta} = \frac{v'}{\Delta \vartheta} = -\nabla p'
\]

Taking the divergence of eqn (9) and assuming the validity of eqn (6) as a means of enforcing continuity through the action of \( \nabla p' \) we obtain

\[
\nabla^2 p' = \frac{1}{\Delta \vartheta} \nabla \cdot v^*
\]

i.e., the Poisson equation for the pressure correction \( p' \). Finally, we use \( p' \) to compute the pressure field

\[
p^{n+1} = p^* + p'
\]
and the velocity correction \( v' \), obtained from eqn (9), to compute the velocity field

\[
v^{n+1} = v^* + v'
\]  

(12)

When the velocity and pressure fields have been computed, we can solve the energy equation by means of the Crank-Nicolson scheme

\[
\frac{t^{n+1} - t^n}{\Delta \vartheta} + \frac{1}{2} v^n \cdot \nabla (t^{n+1} + t^n) = \frac{1}{2Pe} \nabla^2 (t^{n+1} + t^n)
\]  

(13)

before moving to the next step.

The algorithm described here has several features in common with the SIMPLER method [8], even if it must be underlined that the pressure equations are now derived from differential, and not discretized, approximations of the momentum and continuity equations. As a consequence, when eqn (7) is discretized it is not possible to ensure that its terms are consistent with the corresponding pressure terms in the space discretized version of the momentum equation (8). Therefore, as shown in Reference 5, correction (11) is necessary since local continuity is not satisfied exactly (even if global mass conservation can be granted). On the other hand, by maintaining this correction, eqn (8) to (12) yield a SIMPLE-like algorithm which can be conveniently utilized, for example, with low Reynolds number flows [8].

An advantage of our algorithm with respect to the original SIMPLE and SIMPLER methods, is its transportability. In fact, by deriving the pressure equations from continuous approximations we arrive at continuous Poisson equations for pressure. These equations can be solved without using shape functions of lower order in the finite element context, or staggered grids in the control volume context. Furthermore, boundary conditions for pressure are simplified, since eqns (5) and (9) represent equilibrium conditions in the absence of both viscous and convective forces. Thus, zero normal derivatives of pressure, i.e., the natural boundary conditions, can be utilized throughout, except at the reference point(s) where pressure levels are set by assuming \( p^* = 0 \) or \( p' = 0 \). The above physical explanation has been justified analytically in Reference 5, under the condition that global mass conservation is enforced. In our procedure this is achieved by means of a simple correction procedure applied to the velocity profile on the outflow boundary, since velocities are prescribed on all the other boundaries.
Numerical Results

The finite element formulation corresponding to the solution strategy outlined before is based on the Bubnov-Galerkin method with equal order shape functions. Standard definitions, not reported here for the sake of brevity, apply to matrix and vector entries in the discretized equations [5, 7]. The advective boundary conditions for velocity components and temperature have been implemented as the advective boundary condition for temperature in Reference 4. The only difference concerns the boundary mass matrices which, in this study, have been lumped.

The numerical example deals with the transient solution of the mixed convection problem illustrated in Figs. 1 (a) and 1 (b) with $\text{Re} = 10$, $\text{Pe} = 20/3$ and $\text{Fr} = 1/150$, as in Reference 2. The finite element mesh, shown in Fig. 1 (c), has 8x20 nine-node isoparametric elements, yielding a total of 697 nodes. The time steps utilized in the time integration are $\Delta \theta = (\Delta \theta)' = 0.001$ and the calculations were continued until the spatially averaged Nusselt numbers differed less than 0.1% between two successive cycles [4]. The results obtained are illustrated in Figs. 2 to 4, where we represent the behaviour of streamlines, pressure and temperature contours during one complete period.

Figure 2: Periodic evolution of streamlines at time intervals of a quarter of a period. Streamline values from -0.67 to 1.67, step 0.13.
Figure 3: Periodic evolution of pressure contours at time intervals of a quarter of a period. Contour values from -57 to 33, step 6.0.

Figure 4: Periodic evolution of temperature contours at time intervals of a quarter of a period. Contour values from 0 to 1, step 0.1.
As can be seen, the flow consists of travelling roll cells whose axes are normal to the \((x, y)\) plane. A start-up region of length between 1 and 1.5 is evident at the left side of each plot. The advective conditions utilized here lead to very little reflection and distortion at the outflow, even with a very short mesh. All the representative variables show a periodic behaviour with time which can be plotted to allow the evaluation of the period. As an example, in Fig. 5 we illustrate the temperature vs. time curve with reference to the centre of the outflow boundary. The excursion of the temperature, and of all the other variables involved, is not symmetric because the roll motion is superimposed on the main flow.

Figure 5: Time-temperature curve at the centre of the outflow boundary.

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Figure 6: Longitudinal pressure distribution on the top wall with the superimposed Poiseuille pressure field.
Table 1: Comparison of numerical results.

<table>
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<tr>
<th>Parameter</th>
<th>Reference 2</th>
<th>Reference 4</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>1.3319</td>
<td>1.273</td>
<td>1.314</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>1.4465</td>
<td>1.45</td>
<td>1.465</td>
</tr>
<tr>
<td>( \overline{Nu} )</td>
<td>2.5583</td>
<td>2.574</td>
<td>2.5753</td>
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<td>( u_{\text{max}} )</td>
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<td>=</td>
<td>4.5648</td>
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<tr>
<td>( u_{\text{min}} )</td>
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<td>=</td>
<td>-3.0036</td>
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<tr>
<td>( v_{\text{max}} )</td>
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<td>=</td>
<td>5.0709</td>
</tr>
<tr>
<td>( v_{\text{min}} )</td>
<td>-5.0587</td>
<td>=</td>
<td>-5.0709</td>
</tr>
</tbody>
</table>

From the pressure solution, one can evaluate the pressure at the top wall, finding a distribution which is not spatially periodic. However, as demonstrated in Fig. 6, when we plot the auxiliary variable \( p + (12/Re)x \) as a function of the horizontal coordinate \( x \), we find that a component, periodic in space, is superimposed on the overall linear pressure distribution corresponding to the Poiseuille flow.

All the qualitative results presented in Figs. 2 to 4 are in good agreement with the findings of References 2 and 4. Quantitative comparisons reported in Table 1 concern the period \( \theta \), the wavelength \( \lambda \), the time- and space-averaged Nusselt number \( \overline{Nu} \), and the maximum and minimum velocity components. In the present study the start-up region is not included in the computation of the average Nusselt number since we operate on a domain of length 5, four times shorter than the domains utilized in References 2 and 4. Thus, here \( \overline{Nu} \) refers to the length \( 1 \leq x \leq 5 \).

As can be seen, the accuracy is good, even if the domain referred to is short and the mesh utilized is relatively coarse. It is also interesting to note that only the vertical velocity appears to be antisymmetric, with \( v_{\text{max}} = -v_{\text{min}} \), while the horizontal velocity is affected by the main flow in \( x \) direction.

Conclusions

A finite element solution has been presented for the transient laminar flow in a two-dimensional, open-ended, horizontal channel heated from below under conditions which result in a thermoconvective instability. To avoid solving the governing equations in a semi-infinite region, an artificial outflow boundary has been placed in a strongly recirculating zone at a short distance from the inlet. Thanks to the advective boundary condi-
tions employed, the travelling waves leave the domain with very little reflection and distortion. Despite the relative coarse mesh utilized, the numerical results are in good agreement with the benchmark solutions.

References


