The analysis of bubble cavitation inception in turbulent flow around a hydrofoil

L. Wilczyński

Ship Propeller Department, Institute of Fluid Flow Machinery of the Polish Academy of Sciences, Fiszera 14, 80-952 Gdańsk, Poland

Abstract

The paper dwells on the physical assumptions and the mathematical description of the model of the bubble cavitation inception in the flow around a hydrofoil. The objective of the model is to anticipate the risk of the cavitation inception when the flow around the hydrofoil is turbulent. The methods of the stochastic processes theory have been applied in order to determine the time evolution of the bubble radius probability distribution. The numerical application of the model is discussed. The methodology of combined experimental and theoretical research of the bubble cavitation inception in the flow around the arbitrary hydrofoil is proposed.

1 Introduction

Unsteady cavitation is extremely dangerous for the operating fluid-flow facilities, such as ship propellers, stern blades, pumps, turbines, valves and chemical installation pipes because it causes the intense erosion of their solid boundaries and significant decrease of their efficiency. It should be also mentioned that the high level of the hydroacoustic emission generated by the oscillating bubbles during the initial phase of cavitation is the disadvantageous and harmful factor. On the other hand bubble cavitation may be an advantageous effect, for example when appears during the distillation process. Thus the cavitation inception can not be regarded as an indifferent phenomenon from the technical point of view. Additionally the analysis and prediction of the bubbles behavior in the turbulent flow zone creates a very complicated theoretical and experimental problem. The outline of the efforts made in order to solve that problem has been given by Rood.
The majority of the researchers involved in cavitation inception explanation agrees that in order to initialize the rapid vapor bubble expansion the coincidence of two factors is necessary: the first one is the appearance of the sufficient pressure drop and the second one is the presence of a cavitation nucleus during the appropriate time interval in a particular volume of the liquid. Figure 1 presents the schematic view of the typical cavitation inception pattern in the flow around the hydrofoil.

In case the cavitation incides in the turbulent flow both mentioned factors should be understood as the random variables. Consequently the course of each bubble volume variation represents the realisation of a certain stochastic process. The fundamental objective of the presented approach to the cavitation inception analysis is the proper construction of the process governing the time variations of the probability density function of the bubble radius distribution with respect to the flow parameters and the initial nuclei distribution.

Figure 1: A schematic diagram of the initial stage of cavitation - cavitation bubbles appear inside the zone of the turbulent pressure fluctuations on the suction side of the hydrofoil.

2 Physical model and its mathematical description

2.1 General considerations

The method of prediction of cavitation bubbles behavior in turbulent flow field is based on the Lagrangian approach and the so called ‘pdf method’ applied simultaneously to the bubbles population. The model represents the description of time variations of the spherical bubble radius \( r = r(t) \). The trajectories of the cavitation nuclei intersect the region covered by the random pressure fluctuations. The basic aim of the model is to predict the probability of the realisation of the certain class of the \( r = r(t) \) functions with respect to the flow parameters and the hydrofoil geometry.

The stochastic process which realisations are the courses of the pressure variation at any point of the turbulent flow region is the continuous process.
the purpose of the cavitation inception modelling and additionally in order to enable the numerical algorithm construction a discrete time process is assumed. The physical meaning of this assumption is that the pressure changes in the bubble neighborhood are sudden and a certain time interval divides the successive pressure fluctuations. It can be accounted for that the inertia of the fluid surrounding each bubble and the surface tension interaction cause the time integration of the pressure variations.

As usual in such cases it is assumed that the variations of the nucleus and bubble radius being the consequence of the pressure fluctuations are governed by the Rayleigh-Plesset equation:

\[ r \cdot \ddot{r} + \frac{3}{2} r^2 = \frac{1}{\rho_1} \left( p_i - p_o - \frac{2\sigma}{r} - 4\mu_1 \frac{\dot{r}}{r} \right) \]  

where:

- \( r \) is the bubble radius (the dots denote the time derivatives),
- \( p \) is the pressure (subscripts 'i' and 'o' denote the location inside and outside of the bubble respectively),
- \( \rho, \sigma, \mu \), denote density, surface tension and the viscosity coefficient (the subscript 'l' refers to the liquid surrounding the bubble).

As it has been reported by Daily & Johnson\(^1\), O'Hern & Acosta\(^3\), Katz\(^5\) and Shen & Peterson\(^7\) the cavitation nuclei activate inside the turbulent zone of the flow and the cavitation incides within the region of intensive velocity and pressure fluctuations. For the purpose of cavitation inception risk anticipation, such a brief location seems to be quite suitable and the knowledge whether or not the inception takes place with respect to the fluctuations of the flow parameters is more important and valuable. Moreover the location of the cavitation inception region in relation to the hydrofoil surface is constantly changing as far as the phenomenon is caused by the turbulent pressure fluctuations. The coordinate of each individual cavitation nuclei or bubble is the random variable in case it moves through turbulent region of the flow. The reconstruction of each bubble trajectory and then the determination of the pressure at any point the trajectory could make the model ineffective. The stochastic nature of the phenomenon allows for the different approach.

Summarising the application of the stochastic processes theory methods for the description of the cavitation inception in the flow around a hydrofoil is based on the following assumptions:

- the average value of the pressure does not lead to the cavitation inception at any point of the considered flow region,
- the local pressure drops causing the cavitation inception are the consequence of the random fluctuations,
- the appearance of each individual cavitation nucleus in the inception zone is the random event.

The modelled phenomenon in case of the flow around the hydrofoil can be presented as follows:
2.2 The description of the model

The model of the bubble cavitation inception is based on the following assumptions (some of them overlap, but it is necessary to know as complete as possible list of the potential directions of the model improvement):

- the bubbles are of the spherical shape described by the value of its radius; the bubble response to the variations of external pressure are governed by the Rayleigh-Plesset equation (1);
- the distance between any two bubbles in the initial phase of the cavitation is such that the consideration of the mutual bubble's interaction is not necessary;
- the pressure distribution on the bubble surface is uniform (the bubbles remain spherical during the entire process);
- the changes of the pressure in the liquid surrounding the bubble, being the consequence of fluctuations are sudden;
- the changes of the bubbles' volume does not affect the external pressure field;
- the bubbles are examined separately; the frame of reference coincides always with the bubble center.

From the above assumptions it arises that the stochastic process describing the changes of the individual bubble radius is determined by the following equation:

\[ r_i = r_0 + \sum_{i=1}^{n} \Delta r_i \]  

(2)

where \( n \) denotes the random variable representing the number of pressure fluctuations acting on the bubble during the time \( t \) and \( r_0 \) denotes the random variable with pdf corresponding to the initial bubble radius distribution. The quantity \( \Delta r_i \) expresses the change of the bubble radius caused by the pressure fluctuation occurring at the time \( t_i \). The change \( \Delta r_i \) takes place during the interval \( \Delta t_{i+1} = t_{i+1} - t_i \), while the external pressure is assumed to remain
constant. The interval T can be stated as the total time which the bubble spends in the fluctuation covered region. Thus the sequence of moments denoted by the \( t_i \) corresponds to the sequence of the succeeding pressure fluctuations and the following condition is fulfilled:

\[
\sum_{i=1}^{n} \Delta t_i = t, \quad 0 < t < T
\]  

(3)

The probability of the inclusion of the bubble radius within a certain range is the function of time \( t \).

According to the theory of stochastic processes terminology the outlined process can be classified as the one-dimensional, limited and randomized random walk. The bubble radius and the coordinate of the particle performing random walk are the analogical quantities for the mentioned processes. The extreme pressure values in the considered flow area determine the limitations for the bubble radius variation. The randomization of the process consists in that the waiting time \( \tau_i = t_i - t_{i-1} \) dividing the succeeding steps of the process remains a random variable. The additional problem is that the pdfs of the successive steps magnitude alter during the process. This feature of the process corresponds to the variation of the successive pressure fluctuation pdfs.

The following condition can be defined as the cavitation inception:

\[
r_i < r_c < r_{\text{max}} \quad \text{for} \quad t = t_0
\]  

(4)

In case of \( r_{\text{max}} < r_c \) there is no cavitation. The condition \( r_i < r_c \) for \( t = t_0 \) indicates that there are only cavitation nuclei distributed in the liquid initially.

The following definitions can be applied in order to establish properly the stochastic process modelling the bubble radius variation:

let \( X_{t_0}, X_{t_1}, X_{t_2}, \ldots \) denote the sequence of random variables and

\[
S_{t_n} = \sum_{i=0}^{n} X_{t_i}
\]  

(5)

then:

**definition 1.** for a certain sequence \( \{S_{t_i}\} \) at the moment \( t_n \) the record value appears if and only if

\[
S_{t_n} > S_{t_j}, \quad j = 0, 1, 2, \ldots, n - 1;
\]  

(6)

**definition 2.** let \( I \) denotes arbitrary opened range \((a,b)\) on an axis, The event \( S_{t_n} \in I \) is called the visit in the range \( I \) at the moment \( t_n \).

According to the definition 2 the visit of the particle performing the random walk within the interval \( I_c = \{r: r > r_c\} \) stands for the cavitation inception. The probability of the visit within the range \( I_c \) depends in general on the pdf of the single step length, initial radius \( r_0 \) and the number of steps performed. In order to describe the initial phase of the cavitation during a particular time interval \( T \) the following probability distributions are defined:
the probability distribution of the record values for the sequence given by (2). If \( r^*_n \) denotes the occurrence of the record value in the sequence \( \{ r_n \} \) then the distribution:

\[
Pr(r^*_n, t_n) = Pr\{r^*_n < r, t_n < t\} \tag{7}
\]

enables the determination of the cavitation inception probability \( Pr_{ci} \) in the following way:

\[
Pr_{ci} = 1 - Pr\{r^*_n < r_e, t_n > T | t_{n-1} \leq T\}. \tag{8}
\]

- the probability distribution of the number of the visits within the range \( I_c \)

\[
Pr_{cf} = \Pr\{N_{t_k} = k\}; \quad k = 0, 1, \ldots, n. \tag{9}
\]

The event denoted by \( \{ N_{t_k} = k \} \) specifies that the number \( k \) of \( n \) steps of the random walk fulfills the condition \( r \in I_c \). The distribution (9) enables the determination of the fraction of the total pressure fluctuations number leading to the cavitation inception.

- the probability distribution of the time spent in the range \( I_c \). Let \( i \) denotes the number of the successive steps of random walk, then \( \Delta t_i \) denotes the time dividing two successive pressure fluctuations. Random variables \( \Delta t_i \) fulfil the condition:

\[
\sum_{i=1}^{n-1} \Delta t_i = T. \tag{10}
\]

If the number of the steps included in \( I_c \) is \( k \) then the fraction:

\[
Pr_{cd} = \frac{\sum_{j=1}^{k} \Delta t_j^c}{T} \tag{11}
\]

specifies the measure of the duration of the cavitation initial phase. The quantities \( \Delta t_j^c, \quad j = 1, \ldots, k \) denote the time intervals corresponding to the steps performed within the range \( I_c \).

Because the bubbles interaction is neglected the variation of the individual bubble radius is considered. That operation is repeated for the set of the statistically independent bubbles described by the initial radius distribution. The course of the bubble radius change is modelled as follows: let \( r_0 \) denotes the initial bubble radius corresponding to the time \( t_0 \). The moment \( t_0 \) is the moment the bubble enters the fluctuation covered region. Let’s assume \( \dot{r}_0(t_0) = 0 \). Beginning from the moment \( t_0 \), until the moment \( t_0 + T \) the pressure \( p \) in the bubble neighbourhood can be decomposed to the sum of the average value \( \bar{p} \) and the fluctuation \( \delta p \):

\[
p = \bar{p} + \delta p. \tag{12}
\]
The bubble forced by the pressure $p$ at the moment $t_0$ tends to the equilibrium state described by the condition $f(t) = 0$. Within the analysed flow region the pressure is given by certain probability distribution. Thus the bubble tends to different equilibrium states with different probabilities. The set of permissible equilibrium states is described by different values of the bubble equilibrium radius $r^E$, for $f(t) = 0, t \geq t^E$ and different time intervals $t^E$ necessary to attain each of them. As it arises from the equation (1) describing the evolution of the bubble, the values $r^E$ and $t^E$ depend on the pressure $p(t_0)$, initial radius value $r_0$ and $f(t_0)$. Thus the bubble characterised at the time $t < t_0$ by the radius $r_0$, after the action of the first pressure fluctuation at the moment $t_0$, namely at the time $t > t_0$ can be described by one of the states that form the ensemble $S_1$. Each of the state belonging to the ensemble $S_1$ is specified at any moment $t > t_0$ by the values of the actual bubble radius $r(t)$ and its derivative $f(t)$. The set of states $S_1$ evaluates to the certain set of equilibrium states $S^E_1$. Each bubble state included in $S^E_1$ fulfils the condition $f(t) = 0$. As the time after the action of the pressure fluctuation $p(t_0)$ passes by, the common part of the ensembles $S_1$ and $S^E_1$ increases. Basing on the knowledge of the pdf of the random variable $p(t_0)$ and the value of $r_0$, the time $t^E_1$ necessary to achieve $S^E_1$ can be estimated. Summarising time interval $t^E_1$ is characterised by the following property: $S_1 \rightarrow S^E_1$ for $t_1 \rightarrow t^E_1$.

The pdf specified for the ensemble $S_1$ enables the determination of the cavitation inception probability defined above by the equation (8) at any particular moment $t > t_0$. The limiting value of the bubble radius $r_c$ occurring in (8) can be specified arbitrary with respect to the phenomenon scale.

It is obvious that, usually more than one pressure fluctuation act on the bubble. Thus at the moment $t_1 > t_0$ a sudden change of pressure appears. The difference between the moments $t_0$ and $t_1$ consists in that the pressure $p(t_0)$ at the moment $t_0$ modifies the bubble described by the certain state: $\{r(t_0) = r_0, f(t_0) = 0\}$, however the pressure $p(t_1)$ at the moment $t_1$ acts on the bubble specified by the set of states $S_1|_{t=t_1}$. If $t_1 < t^E_1$, then the additional difference is that particular fraction of states included in $S_1|_{t=t_1}$ fulfils the condition $f(t_1) \neq 0$. The ensemble $S^E_2$ is determined by the pdf of the random variable $p(t_1)$ and the probability distribution specified for the actually (at the moment $t_1$) accomplished set of the bubble states. Each state forming $S^E_2$ is
obtained by a transformation (1) executed on $S_{t_{1}}^{t_{2}}$ with respect to the conditions specified by the $p(t_{1})$ pdf.

The pressure fluctuation $\delta p(t_{2})$ happening at the moment $t_{2} > t_{1}$ modifies the set of possible bubble states $S_{t_{1}}^{t_{2}}$. That fluctuation generates the probabilistic copy of the process realization for $t_{1} < t < t_{2}$.

The defined above quantities in the equations (8), (9) and (11):

- $Pr_{ci}$ - probability if the cavitation inception,
- $Pr_{cf}$ - probability of the occurrence of certain number of pressure fluctuations destabilising a bubble,
- $Pr_{cd}$ - probability of a particular duration of the cavitation initial phase,

form the probabilistic description of the cavitation inception. Above quantities can be determined with the help of the set of the two-dimensional probability distributions $Pr(r, \dot{r}; t)$ specified for the ensembles $S_{t}$ ($t$ is the parameter). It is possible because the distribution $Pr(r, \dot{r}; t)$ defined at any particular moment $t_{0} < t < t_{0} + T$ for the ensemble $S_{t}$ enables the determination of the probability of any event defined in the following way: $\{r_{1} < r < r_{2}, \dot{r}_{1} < \dot{r} < \dot{r}_{2}\}$.

As it has been written above the probability distribution $Pr(\delta p, t)$ is being modified at the moments that form a sequence $\{t_{i}\}, i = 0, 1, ..., n$ corresponding to the pressure fluctuations. The time intervals $\Delta t_{i}$ dividing the successive pressure fluctuations form a sequence of random variables as well as the successive pressure fluctuations $\delta p_{i}$. The universal determination of the relation between the parameters of the pdfs of the random variables occurring in the two above sequences and the macroscopic parameters of an arbitrary turbulent flow is impossible. The establishment of ‘a priori’ extensive assumptions concerning the stochastic nature of the phenomenon is the most common way of solving such problems.

The neglection of the bubbles’ trajectories investigation during the cavitation inception arises from the assumption of the ergodicity of the process of the bubbles motion across the turbulent region. The conclusion from that assumption can be formulated as follows: all possible trajectories in the random varying pressure field have the equal probability and after a sufficient time all of them will be covered.

Consequently the Poisson process can be assumed to describe the successive moments of pressure fluctuations occurrence in the bubble neighborhood. The above conclusion indicates that the pdf of the time interval $\Delta t_{i}$ dividing the successive pressure fluctuations is exponential and can be expressed as follows:

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

(13)
The average value of the probability distribution (13) \( \langle X \rangle = \frac{1}{\alpha} \) comes from the fluctuations frequency spectrum. The determination of the turbulent pressure fluctuation spectrum enables to specify the average values \( \langle E \Delta t \rangle \) as the weighted averages of the reciprocals of the frequencies occurring in the spectrum.

The initial bubble radius distribution can be also assumed exponential as the consequence of the lack of the nuclei mutual interaction. The results of the experimental research reported by Gindroz & Briancon-Marjollet\(^2\) and Rood\(^7\) confirm that assumption.

The pdf of the random variable \( \delta p_1 \) coincides usually with the gaussian pdf:

\[
f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right).
\]  

According to (14) the average value \( m \) equals to zero and the variance \( \sigma^2 \) depends on the turbulence intensity. The local value of the parameter \( \sigma \) of the \( \delta p_1 \) distribution depends on the turbulent fluctuations spectrum.

Because of the nonlinearity of the equation (1) transforming the pressure fluctuations probability distributions into the probability distributions defined on the sets of the bubble states \( S^i \), the analytical solution of the stated problem seems to be impossible.

### 3 Numerical application of the model

The numerical algorithm based on the above model, enabling the anticipation of the influence of the random pressure fluctuations on the cavitation bubble radius pdf evolution has been constructed. The detailed description of the structure of the algorithm and the numerical simulations was reported\(^9\). The conclusions arising from the numerical simulations performed can be summarised as follows:

- the probability of the inception probability approaches to the certain value,
- the increase of the magnitude of the pressure fluctuations causes the decrease of the limit value of the inception probability,
- the increase of the initial average bubble radius and the pressure fluctuations average duration time cause the increase of the inception risk.

The detailed experimental validation of the described above theoretical model requires the simultaneous measurements of the pressure distribution on the hydrofoil surface side and the cavitation nuclei distribution in the water. Nevertheless the qualitative comparison of the simulation results and the experimental research results\(^4\) have proved the satisfactory agreement between them.

### 4 Conclusion

The bubble cavitation inception appears in the unsteady flow around the hydrofoil when the initial nuclei distribution and the pressure fluctuations
parameters vary within the relatively narrow range. Thus the detailed analysis of that phenomenon might enable the creation of the specific ‘map’ of the cavitation risk which coordinates would be the described parameters. Such a map could be very useful for anticipation of the probability cavitation inception in the flow around the arbitrary hydrofoil of known geometry. That aim requires thorough both experimental and theoretical research. The ideas presented in this paper may serve as a first step on the way towards its achievement.

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