Traveling waves in multi-phase flows
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Abstract

Equations of motion based on the theory of interacting and interpenetrating continua are used to study pattern forming instabilities in two-dimensional dispersed two-phase flows such as a suspension of particles in a fluidized bed and a bubbly gas-liquid flow. By making use of analytical and numerical bifurcation techniques it is shown that the main instability sequence in a vertically suspended configuration consists of the transition from a homogeneous flow through a one-dimensional concentration wave train to a two-dimensional traveling wave.

1 Introduction

We consider particle suspensions in a fluidized bed which may be described by two interacting and interpenetrating Newtonian continua. Such flows have a wide variety of applications in industry but are still poorly understood despite a great amount of experimental and theoretical work. On the theoretical side, it has been known for some time that the ideal state of uniform fluidization is usually unstable to small disturbances, and that this primary instability leads to vertically traveling plane wave trains which may be related to slug flow in narrow channels. Experimental observations in wider beds, however, point toward the occurrence of two- or three-dimensional non-uniformities like rising bubbles in gas-fluidized beds or planar traveling waves developing a transverse structure in liquid-fluidized beds. There has been an ongoing speculation about the origin of such voidage non-uniformities. Previous direct numerical simulations of gas-fluidized beds have revealed bubble-like structures, and in one particular study it was shown that spatially-periodic two-dimensional...
solutions result from a secondary instability of the planar wave train which bifurcates from the uniform state. But neither the origin of this instability nor the differences between gas- and liquid-fluidized beds could be cleared up. Therefore, we have combined analytical and numerical methods to prove and clarify the existence and nature of secondary traveling waves and to contrast gas- and liquid-fluidized beds.

The same type of analysis can be applied to the “downward” fluidization of light particles by a heavy fluid, or to the flow of gas bubbles through a liquid as the scale-up behaviour and the modeling is very similar, regardless of the different parameter ranges.

2 Model equations

We study the following set of mass and momentum balance equations

\begin{align}
-\partial_t \phi + \text{div} \left[ (1 - \phi) \mathbf{v} \right] &= 0, \quad \text{(dispersed phase)} \\
\partial_t \phi + \text{div} \left( \phi \mathbf{u} \right) &= 0, \quad \text{(continuous phase)} \\
F(1 - \phi)(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) &= -(1 - \phi) \mathbf{k} + B(\phi)(\mathbf{u} - \mathbf{v}) - G(\phi)\nabla \phi \\
&\quad - (1 - \phi) \nabla p + \frac{F}{R}(\Delta + \kappa \nabla \nabla \cdot \mathbf{v}), \\
F\delta \phi(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) &= -\delta \mathbf{k} - B(\phi)(\mathbf{u} - \mathbf{v}) - \phi \nabla p \\
&\quad + \frac{F}{R}(\Delta + \bar{\kappa} \nabla \nabla \cdot \mathbf{u}),
\end{align}

where $\phi \in (0, 1)$ and $\mathbf{u}$ denote the volume fraction (“voidage”) and local velocity, respectively, of the continuous (fluid) phase, $\mathbf{v}$ represents the velocity of the dispersed phase (particles or gas bubbles), $p$ is an effective fluid pressure, $G(\phi)\nabla \phi$, with $G(\phi) < 0$, represents a “porosity” gradient term stemming from interactions within the dispersed phase, and $B$ is a drag force coefficient of the form

$$B(\phi) = |1 - \delta| \frac{1 - \phi}{\phi^n} \phi_0^{n+1}. \quad (5)$$

Here, $\phi_0$ is the voidage value of the base state of uniform fluidization, where one phase is at rest and the other is moving steadily up- or downwards with velocity $s_0 = |1 - \delta| \phi_0^{n+1} \rho_d g/(9 \mu_c/(2r^2))$. This is the well-known semi-empirical Richardson-Zaki law, with an exponent $n$ in the range $2 \sim 4$. Using $s_0$ and a typical particle or gas bubble radius $r$ as scaling parameters leads to a Froude number $F = s_0^2/(gr)$ and a Reynolds number $R = \rho_d r s_0 / \mu_d$; furthermore, it is convenient to introduce the relations of densities, $\delta = \rho_c/\rho_d$, and viscosity coefficients, $\nu = \mu_c/\mu_d$, as well as other viscosity related coefficients $\kappa$ and $\bar{\kappa}$. Within this frame the basic homogeneous flow is given by

$$\nabla p_0 = -[1 - \phi_0(1 - \delta)] \mathbf{k}, \quad \mathbf{u}_0 - \mathbf{v}_0 = \text{sgn}(1 - \delta) \cdot \mathbf{k}. \quad (6)$$
In the case of the “usual” fluidization of relatively heavy particles with a gas or a liquid we have $\delta < 1$, and the base state is given by macroscopically quiescent particles and a uniformly upwards moving fluid, i.e., $u_0 = k, v_0 = 0$. If we consider the inverse fluidization of light particles like small polystyrene spheres or the upward flow of tiny gas bubbles through a liquid at rest, $\delta > 1$ and, hence, $u_0 - v_0 = -k$. For a common treatment of all these cases we put

$$v_0 = 0, \quad u_0 = u_0 k, \quad \text{i.e.,} \quad u_0 = \text{sgn}(1 - \delta),$$

so that in the second case the fluid is moving downwards with unit velocity.

The reader will notice that we neglected the virtual mass and lift forces. While the former is not crucial for the analysis that follows, incorporating the latter would make it significantly more difficult.

### 3 Primary instabilities

#### 3.1 Linear stability

The linear stability of the base state is determined by the equation

$$(A\partial_t^2 + 2u_0 C\partial_t \partial_z + C\partial_z^2 + D\partial_z + E\partial_t - M\Delta - u_0 H\Delta\partial_z - J\Delta\partial_t)\phi = 0,$$

with positive coefficients

$$A = \phi_0 + C, \quad C = \delta(1 - \phi_0), \quad E = \frac{B_0}{F\phi_0(1 - \phi_0)} = \frac{|1 - \delta|}{F},$$

$$FD = u_0 \left(\frac{B_0}{\phi_0} - B_0\right) + (1 - \delta)(1 - 2\phi_0) = (1 - \delta)(n + 2)(1 - \phi_0),$$

$$M = -\frac{\phi_0 G_0}{F}, \quad H = \frac{v(1 + \kappa)(1 - \phi_0)}{R\phi_0}, \quad J = \frac{(1 + \kappa)\phi_0}{R(1 - \phi_0)} + H.$$

It is convenient to define the parameters

$$m = \frac{M}{A}, \quad c = \frac{C}{A}, \quad h = \frac{H}{J}, \quad d = \frac{u_0 D}{E} = (n + 2)(1 - \phi_0),$$

and the function

$$f(s) = m - c(1 - c) - (s - c)^2,$$

so that the stability condition for perturbations of longitudinal and transverse wavenumbers $\lambda$ and $k$, respectively, can be written in the form

$$mk^2 + \lambda^2 \cdot f(q(\lambda^2 + k^2)) > 0, \quad \text{where} \quad q(s) = \frac{u_0 D + Hs}{E + Js}.$$
It follows that the base state is linearly stable if the two conditions \( f(d) > 0, \ f(h) > 0 \) are met. Depending on the parameters \( h, \ d, \) and \( c, \) one of \( f(d) > 0 \) or \( f(h) > 0 \) is sufficient for stability due to the relations

\[
f(h) > f(d) \iff d > h, \ h + d > 2c \text{ or } d < h, \ h + d < 2c.
\] (13)

In order to understand these inequalities in terms of \( \delta \) and \( \phi_0, \) we note that \( h + d > 2c \) is satisfied for a density ratio \( \delta < 1, \) while \( h + d < 2c \) is only possible if \( \delta > 1 \) and \( \phi_0 > n\delta/[(\delta - 1)(n + 2)] + O(h). \) Moreover, if the effective viscosity of the dispersed phase is larger than that of the continuous phase, i.e. \( \nu < 1, \) one can argue that \( h < d; \) this together with the condition \( h + d > 2c \) implies \( f(h) > f(d). \) This was the case dealt with previously\(^8\), but now we have to consider the possibilities \( d < h \) and/or \( h + d < 2c \) also. Obviously \( h < 1, \) hence a condition \( d < h \) restricts the basic voidage to values \( \phi_0 > 1 - h/(n + 2) > (n + 1)/(n + 2), \) which means that the dispersed phase would be rather dilute in such a case. On the other hand, \( d - 2c \) is positive for all \( \phi_0 \in (0, 1) \) if \( \delta \leq (n + 2)/2, \) and for all \( \phi_0 < n\delta/[(\delta - 1)(n + 2)] \) if \( \delta > (n + 2)/2; \) hence a voidage regime \( \phi_0 \leq n/(n + 2), \) which is typical for fluidized beds, allows for \( d > h, \ f(h) > f(d) \) only.

### 3.2 Primary bifurcations

If the base state is unstable, a bifurcation to a family of traveling waves occurs. This can be shown by transforming eqns. (1)-(4) to a moving coordinate system with \( z \rightarrow z - u_0\omega t \) and looking for stationary solutions in the new system; the wave velocity \( \omega \) serves as the bifurcation parameter. Quasi-stationary periodic solutions of eqn. (8) are then found by incorporating the replacements \( \partial_t \rightarrow -u_0\omega \partial_z, \ \partial_z \rightarrow i\lambda, \ \partial_y \rightarrow ik, \) which leads to the conditions

\[
f(\omega)\lambda^2 + mk^2 = 0, \quad E(\omega - d) + J(\omega - h)(\lambda^2 + k^2) = 0.
\] (14)

Note that (14) is the same for both \( \delta > 1 \) and \( \delta < 1; \) however, the waves travel upwards for \( \delta < 1 \) and downwards for \( \delta > 1 \) (relative to the dispersed phase). The second condition in (14) requires \( \omega \in (\min\{h, d\}, \max\{h, d\}) , \) which together with \( f(\omega) \leq 0 \) implies linear instability. Eliminating \( k^2 \) or \( \lambda^2, \) respectively, gives the critical wavenumbers as a function of the wave speed,

\[
\frac{J}{E}\lambda^2 = \frac{d - \omega}{\omega - h} \cdot \frac{m}{m - f(\omega)}, \quad \frac{J}{E}k^2 = \frac{d - \omega}{\omega - h} \cdot \frac{-f(\omega)}{m - f(\omega)} \equiv P(\omega).
\] (15)

Since \( m - f(\omega) \geq m - f(c) = c(1 - c) \geq 0, \lambda^2 > 0 \) for wave velocities between \( h \) and \( d, \) hence the longitudinal wavenumber is well defined in
this interval, but becomes singular at $\omega = h$. It is also singular at $\omega = 0$, provided $c = 0$ (due to $\delta = 0$); however, this is outside the “bifurcation domain”, even if $h$ assumes its minimum value of zero (due to $\nu = 0$). Admissible transverse wavenumbers $k$ are those for which $P(\omega) > 0$ (and with wavelength that fits into a given bed width). The somewhat strange behaviour $P(\omega) \to +\infty$ for $\omega \to h^+$, which allows the onset of instabilities with arbitrarily large transverse wavenumbers, can be cured by adding regularizing terms to either (1) or (3), cf.\textsuperscript{16}.

A complete exploration of the bifurcation behaviour should consider both $h < \omega < d$ and $d < \omega < h$. In the present communication we focus on the former case, which is typical in most fluidized suspensions of particles. The possible solutions of (14) for $h < d$ and $\delta < 1$ have been evaluated\textsuperscript{8}; for $\delta > 1$ we get a new solution which was previously excluded, see Fig. 1.

In most cases there exists a parameter value $\omega_0$ between $h$ and $d$ for which $P(\omega_0) = 0$, such that the main instability occurs in the vertical direction and leads to the bifurcation of plane voidage waves $\phi(z - u_0\omega t)$.

Figure 1: $k^2 \sim P(\omega)$ for $h < \omega < d$, $f(h) < 0 < f(d)$ (solid curve); according to (13) this means $h + d < 2c$. Another new case, $f(h) < f(d) < 0$ with $d < c - \sqrt{f(c)}$, gives the broken curve; a curve like this, however, occurs for $\delta < 1$ already and excludes a bifurcation to one-dimensional wave trains\textsuperscript{8}. 

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Other bifurcations along the uniform state give rise to a genuinely two-dimensional vertically traveling wave $\phi(z - u_0\omega t, y)$ as well as a pair of oblique traveling waves $\phi(z - u_0\omega t \pm ky)$; although these are usually already unstable at their onset, they play an important role for the subsequent instability of the plane wave above.

3.3 Plane traveling waves

The oblique and vertically traveling plane waves can be investigated in detail analytically. Because they depend only on the variable $z' = z - u_0\omega t \pm ky$, eqns. (1)-(4) become ordinary differential equations and can actually be reduced to a second-order equation for the voidage, namely (cf.\textsuperscript{17} for the case $u_0 = 1$)

$$f_1\phi'' = f_2\phi'^2 + u_0 f_3\phi' + u_0 f_4,$$

where the $f_i$ are complicated functions of $\phi$, $\omega$, $k^2$ (due to a reflectional symmetry of the system), and the other parameters. They also involve two integration constants which arise from solving eqns. (1) and (2). An easy way to determine these two constants is to claim that the base state is a solution of the integrated equations\textsuperscript{4,17}; this amounts to prescribing the corresponding volumetric fluxes of the two phases and allows for departures of the mean voidage value from that of the uniform state\textsuperscript{18}. In the numerical analysis to be described later, however, we have to deal with the original system of equations and there it is both natural and convenient to fix the mean value of the voidage and, based on the freedom due to Galilean invariance, the mean vertical flux of the dispersed phase\textsuperscript{19}.

Eqn. (16) possesses various steady state and periodic solutions, homoclinic and heteroclinic orbits, and a singularity $\phi_s(\omega)$ for each $\omega \in [0, 1]$, which connects $(\phi, \omega) = (1, 0)$ with $(0, 1)$ monotonically and crosses $\phi = \phi_0$ in $\omega = \delta$. Further details on the case $\nu < 1$, $\delta < 1$ can be found in\textsuperscript{17,19}. Here it is of greater interest to look at the two-dimensional stability of these waves and to determine the structure of possible secondary solutions.

4 Secondary instability

It has been shown recently\textsuperscript{18} that vertically traveling plane wave trains in gas-fluidized beds are unstable to two-dimensional perturbations with the same longitudinal periodicity but large transverse wavelength. The instability sets in when the amplitude of the plane wave has grown to the order of the square of the transverse wavenumber of the perturbation, i.e. $\|\phi(z') - \phi_0\| = O(k^2)$. It can be stationary or oscillatory in nature and has its origin in the interaction between the plane wave and a disturbance packet consisting of four modes. These modes represent initially, i.e. at the primary bifurcation point, four least-stable perturbation modes of the uniform state, two being a pair of two-dimensional “mixed modes” as de-
scribed above, and the other two being pure transverse modes; all develop into mixed modes with an additional transverse structure along the growing plane wave. In principle, any of them can be the dominant one, but the emphasis seems to be on the mixed modes. Preliminary results indicate that the behaviour is very similar in liquid-fluidized beds, whereas the analysis for the cases of downward fluidization and gas-liquid flow has yet to be done.

5 Numerical results

Our numerical approach\textsuperscript{19} combines a pseudospectral discretization procedure with numerical continuation techniques. Since we assume periodicity in both spatial directions and search for fully developed solutions of eqns. (1)-(4) moving at constant speed, all the variables are approximated with a Fourier series in a moving frame. The inner products of the resulting equations with the basis functions are evaluated numerically through the use of the inverse discrete Fourier transform. This leads to a system of differential equations for the time-dependent Fourier coefficients, which is then investigated using the continuation package AUTO\textsuperscript{20}; here, we chose either one of the wavenumbers $\lambda, k$, or the basic voidage $\phi_0$ as the bifurcation parameter. Stationary solutions of this system represent steadily moving periodic waves; their velocity is determined through an additional algebraic equation, which also eliminates the translational invariance of the solution. Further constraints arise from the fact that the volume-averaged value of the (fluid) volume fraction does not change in time, and that only the differences of the averaged velocities of the two phases is determined uniquely. We therefore fix the mean voidage value to $\phi_0$ and require the mean vertical flux of the dispersed phase to vanish; the indeterminacy in the transverse flux is taken care of by admitting horizontally symmetric modes only.

A one-dimensional traveling wave bifurcates from the uniform state subcritically, if $\phi_0$ is slightly larger than a critical value $\phi_c$, but subsequently turns around and becomes stable (with respect to small one-dimensional disturbances of the same periodicity); the bifurcation is supercritical for much larger values of $\phi_0$. If we include two-dimensional disturbances of the same longitudinal periodicity but with transverse wavenumber $k$, we find that the 1-D wave is linearly stable for large values of $k$. As we decrease $k$, loss of stability occurs at a critical value $k_c$, where one real eigenvalue crosses zero and becomes positive, thus signaling the onset of a family of 2-D traveling waves.

The secondary bifurcation can be sub- or supercritical depending on whether the transverse wavenumber is relatively large or small; in any case, however, we obtain stable, high-amplitude solutions which are reminiscent of bubbles in gas-fluidized beds, see Fig. 2. Interesting features of the solution pattern include the possible coexistence of a stable one-
dimensional wave and a fully developed two-dimensional traveling wave, and the possible existence of stable high-amplitude waves even when the base state is linearly stable to perturbations of all length scales. It was also found that the results are qualitatively similar for different choices of model parameters and closure relations, although these do affect the nature of the secondary bifurcation and the shape of the emerging solutions.

Figure 2: Typical bifurcation diagram and high-amplitude solution pattern. The contour plot corresponds to point D in the bifurcation diagram.
6 Conclusions

We have applied analytical and numerical bifurcation methods to evaluate the fate of two-dimensional disturbances of the uniform state of fluidization when the latter is unstable. Two stages of instabilities have been identified, the first leading to periodic plane traveling waves, the second to stable high-amplitude two-dimensional vertically traveling waves. The numerical calculations were restricted to horizontally symmetric modes, so that oblique waves and non-symmetric perturbations of the solutions were excluded. While the analytical and numerical approaches agree on the sequence of instabilities and in particular on the occurrence of the secondary instability and the modes driving it, so far only stationary secondary bifurcations have been found numerically whereas analytical work has shown that an oscillatory instability is also possible.

The above results hold for gas-fluidized beds but current work on liquid-fluidized beds has revealed qualitatively similar behaviour. Nevertheless, quantitative differences are to be expected as direct numerical simulations show that the time and space scales, on which the one- and two-dimensional solutions develop, can be quite different in the two cases.

The described results are in qualitative agreement with experimental observations (a quantitative comparison is beyond the present scope due to the many uncertainties in modeling parameters and closure relations). Therefore, and because the model equations are the same, it is natural to apply the same ideas to the inverse fluidization of light particles and the flow of gas bubbles through a liquid column. The required generalization of the previous considerations to a wider parameter regime has started.

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