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Fundamentals concerning Stokes waves

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Abstract

In the study of water waves, it is well known that linear theory provides a first approximation to the complete wave motion. In order to approach the complete solution more closely, successive approximations using perturbation procedure are usually developed. In his classic paper in 1847, Stokes first established the nonlinear solution for periodic plane waves on deep water. The convergence towards a complete solution does not occur for steeper waves, unless a different perturbation parameter from that of Stokes is chosen. This paper discusses the wave motion when the wave amplitude is large compared to the wavelength. It has been confirmed that the analytical results are valid only for deep water waves. Then the investigation is carried out to obtain the solution of the shallow water wave theory using perturbation method which leads to the solitary wave solution.

1 Introduction

In his 1847 classic paper, Stokes first established the nonlinear solutions for periodic plane waves on deep water. In order to obtain the complete solution, successive approximations using a perturbation procedure was used by Stokes. The nonlinear solutions of the plane waves based on this technique are known as Stokes expansions. A question was, however, raised about the convergence of the Stokes expansions in order to prove the existence of solution representing periodic water waves of permanent form. The problem was eventually solved by Levi-Civita [3], who proved formally that the Stokes expansion for deep water converges provided the wave steepness is very small. Struik [11] extended the proof of Levi-Civita to small-amplitude waves on water of arbitrary, but constant depth. It is worth mentioning here the extensive work of Schwartz [9] on the nonlinear Stokesian waves which have recently received considerable attention. However, this should not detract from the practical advantages of the original expansion procedure of Stokes. In the following, we will discuss the wave motion when the wave amplitude is large compared to wavelength. Then this study is extended to include the shallow water wave theory.

2 Mathematical formulation

The mathematical formulation of finite amplitude wave theory is basically the same as that for small amplitude wave theory. However, in the former case, higher order terms in the free surface are considered important and are retained. It will be clear later that the Stokes expansion method is formally valid under the conditions $\frac{kA}{(kh)^3} << \frac{8}{3}$ for $kh < 1$, and $\frac{H}{L} << 1$. This may be demonstrated by comparing second and first order forms in the expression for the velocity potential to be determined. The above conditions place a severe wave height restriction in shallow water which needs a separate shallow wave expansion procedure and is discussed in the next section. Thus in this section we describe the Stokes expansion procedure valid for deep and intermediate depth water waves.

We have already derived the fundamental equations describing the wave motion in the previous sections. The appropriate equations are reproduced here for convenience only.

Laplace's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{1}$$

Dynamic free surface boundary condition:

$$\eta = -\frac{1}{g}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right\}\right] \quad \text{at} \quad z = \eta \tag{2}$$

Kinematic boundary condition:

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial z} \quad \text{at} \quad z = \eta \tag{3}$$

Bottom boundary condition

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -h \tag{4}$$

To eliminate η between (2) and (3), we take the total derivative of the expressions in (2) with respect to t which yields after rearranging the terms

$$g\frac{d\eta}{dt} = -\frac{d}{dt}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\{(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial z})^2\}\right].$$

Eliminating $\frac{d\eta}{dt}$ by $\frac{\partial\phi}{\partial z}$ from (3) and recalling that $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial z}$, and after a little reduction, we obtain a simple equation in ϕ as

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} + \left[\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial\phi}{\partial z}\frac{\partial}{\partial z}\right]\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] = 0 \quad (5)$$

This equation which was also obtained by Phillips [6] contains the third order term in variable ϕ in (5) and retaining up to the second order term, we obtain

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = -\frac{\partial}{\partial t}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] \quad (6)$$

Using his physical intuition, equation (6) was also obtained by Lighthill [4] in his survey lecture at Imperial College which corresponds to equation (25) of his paper.

It is assumed that ϕ and η may be expressed as the perturbation series in the forms

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots = \phi_l + \phi_q + \cdots \quad (7)$$
$$\eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \ldots = \eta_l + \eta_q + \cdots \quad (8)$$

in which ϵ is the perturbation parameter. The parameter $\epsilon = kA$ is related to the ratio of the wave height to the wavelength, and usually assumed to be small. By substituting equations (7) into Laplace's equation (1) and the boundary condition (4) at the sea bed, and by collecting the terms of order $\epsilon, \epsilon^2, \epsilon^3, \cdots$, we obtain respectively

$$\frac{\partial^2\phi_n}{\partial x^2} + \frac{\partial^2\phi_n}{\partial z^2} = 0 \quad \text{for} \quad n = 1, 2, 3, \cdots \quad (9)$$
$$\frac{\partial\phi_n}{\partial z} = 0 \quad \text{at} \quad z = -h \quad \text{for} \quad n = 1, 2, 3, \cdots \quad (10)$$

This is extremely straightforward. However, the difficulties in surface wave problem arise from the free surface boundary conditions which contain nonlinear terms consisting of products, and which are applied at the unknown surface $z = \eta$ rather than at $z = 0$.

Because the wave surface is not known apriori, it will be advantageous to use the following Taylor expansion for the velocity potential ϕ about $z = 0$.

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Thus

$$\begin{aligned}\phi(x,\eta(x,t),t) &= \phi(x,o,t)+\eta\left(\frac{\partial\phi}{\partial z}\right)_{z=0}+\cdots\\ &= (\epsilon\phi_1+\epsilon^2\phi_2+\cdots)_{z=0}+(\epsilon\eta_1+\epsilon^2\eta_2+\cdots)\\ &\quad\left(\epsilon\frac{\partial\phi_1}{\partial z}+\epsilon^2\frac{\partial\phi_2}{\partial z}+\cdots\right)_{z=0}+\cdots\\ &= \epsilon\phi_1+\epsilon^2\left(\phi_2+\eta_1\frac{\partial\phi_1}{\partial z}\right)+0(\epsilon^3)\quad\text{at}\quad z=0 \qquad (11)\end{aligned}$$

Similarly, expanding by Taylor's theorem, we obtain

$$\frac{\partial\phi}{\partial z} = \epsilon\frac{\partial\phi_1}{\partial z}+\epsilon^2\left[\frac{\partial\phi_2}{\partial z}+\eta_1\frac{\partial}{\partial z}\left(\frac{\partial\phi_1}{\partial z}\right)\right]+\cdots \qquad (12)$$

$$\frac{\partial\phi}{\partial x} = \epsilon\frac{\partial\phi_1}{\partial x}+\epsilon^2\left[\frac{\partial\phi_2}{\partial x}+\eta_1\frac{\partial}{\partial z}\left(\frac{\partial\phi_1}{\partial x}\right)\right]+\cdots \qquad (13)$$

$$\nabla\phi = \epsilon\nabla\phi_1+\epsilon^2\left[\nabla\phi_2+\eta_1\frac{\partial}{\partial z}(\nabla\phi_1)\right]+\cdots \qquad (14)$$

These expressions are to be evaluated at $z=0$.

The terms of order ϵ appearing in the free surface boundary conditions (2) and (3) are given by

$$\eta_1 = -\frac{1}{g}\left(\frac{\partial\phi_1}{\partial t}\right)_{z=0} \qquad (15)$$

$$\frac{\partial^2\phi_1}{\partial t^2}+g\frac{\partial\phi_1}{\partial z}=0\quad\text{at}\quad z=0 \qquad (16)$$

The first order theory which is usually referred to as **Airy wave theory** has the solution as

$$\begin{aligned}\eta_l &= A\sin(kx-\sigma t)\\ \phi_l &= -\frac{Ag}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\cos(kx-\sigma t)\end{aligned} \qquad (17)$$

In the second-order theory (quadratic) we obtain the following equation and boundary conditions.

$$\frac{\partial^2\phi_q}{\partial x^2}+\frac{\partial^2\phi_q}{\partial z^2}=0 \qquad (18)$$

$$\frac{\partial\phi_q}{\partial z}=0\quad\text{at}\quad z=-h \qquad (19)$$

$$\frac{\partial^2\phi_q}{\partial t^2}+g\frac{\partial\phi_q}{\partial z} = -\eta_l\frac{\partial}{\partial z}\left[\frac{\partial^2\phi_l}{\partial t^2}+g\frac{\partial\phi_l}{\partial z}\right]-\frac{\partial}{\partial t}\left[\left(\frac{\partial\phi_l}{\partial x}\right)^2+\left(\frac{\partial\phi_l}{\partial z}\right)^2\right] \qquad (20)$$

$$\eta_q = -\frac{1}{g}\left[\frac{\partial \phi_q}{\partial t} + \eta_l \frac{\partial^2 \phi_l}{\partial z \partial t} + \frac{1}{2}\left\{\left(\frac{\partial \phi_l}{\partial x}\right)^2 + \left(\frac{\partial \phi_l}{\partial z}\right)^2\right\}\right] \tag{21}$$

Now substituting for ϕ_l and η_l from equation (17) into righthand side of equation (20) and after simplifying, we obtain

$$\frac{\partial^2 \phi_q}{\partial t^2} + g\frac{\partial \phi_q}{\partial z} = \frac{3A^2 gk\sigma}{\sinh 2kh} \sin 2(kx - \sigma t) \tag{22}$$

Therefore the solution for ϕ_q can be obtained as

$$\phi_q = -\frac{3A^2\sigma}{8}\frac{\cosh 2k(z+h)}{\sinh^4 kh} \sin 2(kx - \sigma t) \tag{23}$$

and the corresponding free surface elevation, η_q, can be expressed as

$$\eta_q = -\frac{A^2 k}{4}\frac{\cosh kh}{\sinh^3 kh}(2 + \cosh 2kh)\cos 2(kx - \sigma t) - \frac{A^2 k}{2\sinh 2kh} \tag{24}$$

It is to be noted here that η_q contains two parts; one oscillatory $\tilde{\eta}$ and the other one mean $\bar{\eta_q}$ (time independent). Let us define $\eta_q = \tilde{\eta_q} + \bar{\eta_q}$ where

$$\begin{aligned}\tilde{\eta_q} &= -\frac{A^2 k}{4}\frac{\cosh kh}{\sinh^3 kh}(2 + \cosh 2kh)\cos 2(kx - \sigma t)\\ \bar{\eta_q} &= -\frac{A^2 k}{2\sinh 2kh}\end{aligned}$$

3 Discussion of Stokes results

It becomes now clear that the presence of $\bar{\eta_q}$ in (24) causes the η_q to oscillate about the mean free surface in an otherwise undisturbed incident wave. Since $\bar{\eta_q}$ is always negative, becomes more so as the wave enters from the deep water to the shallow water and at the same time the amplitude of oscillations becomes much more pronounced and consequently the wave commences to break. Thus we can argue that the Stokes expansion is only valid for deep water waves. A plot of $\frac{\eta}{A}$ in Figure 1 is depicted with a set of nondimensional paprameters $kA = 0.100$ and $kh = 0.785$. It is to be noted that for the deep water case, the value of the mean water surface is zero whereas for the shallow water case this value is $-\frac{kA^2}{4kh}$.

From the practical consideration, in moderately deep water case, this mean water surface elevation is just a very small additive constant to the main fluctuating parts. Thus inclusion or deletion of this part will not affect the wave profile. For completeness we will retain this part in η.

The velocity potential and water surface displacement to second-order are respectively given by

$$\phi = \phi_l + \phi_q$$

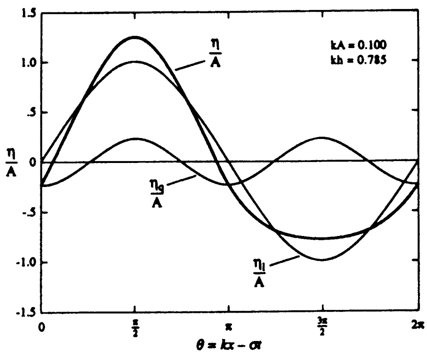

Figure 1: Dimensionless free surface profile of a sine progressive wave

$$
\begin{aligned}
= & -\frac{Ag}{\sigma}\frac{\cosh k(z+h)}{\cosh kh}\cos(kx-\sigma t) \\
& -\frac{3A^2\sigma}{8}\frac{\cosh 2k(z+h)}{\sinh^4 kh}\sin 2(kx-\sigma t)
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
\eta &= \eta_l + \eta_q \\
&= A\sin(kx-\sigma t) - \frac{A^2k}{4}\frac{\cosh kh}{\sinh^3 kh}(2+\cosh 2kh)\cos 2(kx-\sigma t) \\
&- \frac{A^2k}{2\sinh 2kh}
\end{aligned}
\tag{26}
$$

It is worth mentioning here that the error of using these two terms in the perturbation series is not just $0(\epsilon^3)$, but actually $0(\epsilon^4)$.

The dispersion relation between σ and k remains the same

$$\sigma^2 = gk\tanh kh.$$

These solutions are mainly due to Stokes who developed his theory in 1847. The convergence of the series solution was confirmed by Levi-Civita [3] in 1925. However, in the following we give a brief discussion about the convergence of this series.

We know that for the power series for ϕ in terms of ϵ to converge, the ratio of the (n+ 1) th term divided by the nth term must be less than unity as n goes to infinity. Therefore, for the series for ϕ in equation (25) to converge we must have

$$r = \left|\frac{\phi_q}{\phi_l}\right| = \frac{3}{8}\frac{kA\cosh 2kh}{\cosh kh\sinh^3 kh} << 1$$

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For deep water waves, when kh $> \pi$, the asymptotic values of the hyperbolic functions can be substituted to obtain r as $r = 3e^{-2kh}(kA)$. Thus for this case r is very small because kA is a small parameter. It has been found experimentally (Weigel [13]) that the highest value in deep water would occur for $kA = \frac{\pi}{7}$ (for wave for maximum steepness), kh $= \pi$ such that $r = \frac{3\pi}{7}e^{-2\pi} = 0.0025$. Considering this fact, Levi-Civita confirmed that the Stokes perturbation solution is valid up to the second-order term.

In shallow water, $kh < \frac{\pi}{10}$, the asymptotic values of the hyperbolic function can yield the value of r to be $r = \frac{3}{8}\left\{\frac{kA}{(kh)^3}\right\} < 1$. The term within the brackets i.e. $\left\{\frac{kA}{(kh)^3}\right\}$, is defined as the Ursell number. It can be easily seen that in shallow water waves, we have constraints like $kA < \left(\frac{8}{3}\right)(kh)^3$ or $\frac{A}{h} < \left(\frac{8}{3}\right)(kh)^2$, where (kh) is small.

The maximum value that the ratio $\frac{A}{h}$ can attain is $\frac{A}{h} = \left(\frac{8}{300}\right)\pi^2$ for $kh = \frac{\pi}{10}$, or the maximum wave amplitude is about 26% of the water depth. But for the wave amplitude of a breaking wave in shallow water it is almost 40% of the water depth. Therefore, for high values in shallow water, the Stokes expansion is not a good approximation (at least to only second order).

4 Nonlinear long waves in shallow water

In the Stokes second-order theory, the surface profile for small parameter kA (the wave steepness) has been obtained in (26). In very shallow water this Stokes wave profile becomes (see also Struik [11]).

$$\eta(x,t) = A\sin(kx - \sigma t) - \frac{3A^2k}{4(kh)^3}\cos 2(kx - \sigma t) - \frac{kA^2}{4kh}. \tag{27}$$

when $kh \to 0$. We have seen in the previous section that for deep water waves when $kh \to \infty$ this surface profile is

$$\eta(x,t) = A\sin(kx - \sigma t) - \frac{A^2k}{2}\cos 2(kx - \sigma t). \tag{28}$$

From this analysis it is clear that the nonlinear theory of the Stokes expansion for $kA < 1$ is valid for deep water waves only. However, from this nonlinear theory in the case of shallow water waves, we have $\frac{3}{4}\frac{kA}{(kh)^3} < 1$ such that $(kA) < \left(\frac{4}{3}\right)(kh)^3$. This subsequently yields

$$\frac{A}{h} < \frac{4}{3}(kh)^2.$$

Thus for this case both $kh << 1$ and $\frac{A}{h} << 1$ must hold. Let us define these two parameters as

$$\gamma = kh << 1 \quad \text{and} \quad \delta = \frac{A}{h} << 1. \tag{29}$$

Here the second restriction is a much more severe one for many practical problems. Therefore, it is essential to investigate a nonlinear theory of shallow water waves. Historically, two different theories were developed, one by Airy [1] and the second by Boussinesq [2], which led to opposite conclusions. However, these differences were resolved by a fundamental paper by Ursell [12]. In particular, Ursell showed that the ratio,

$$U_r = \frac{\delta}{\gamma^2} = \frac{A/h}{(kh)^2} = \frac{kA}{(kh)^3} \tag{30}$$

which is known as the Ursell number, plays a very important role in deciding the choice of approximation. There are two limiting cases:

(a) Airy's theory for very long waves can be obtained when $\gamma \to 0$ and $\delta = O(1)$ which is also known as the linear wave theory.

(b) Boussinesq theory results when $\delta = O(\gamma^2) < 1$ which produces solitary waves and cnoidal waves.

It is to be noted here that the two parameters $\delta = (\frac{A}{h})$ and $\gamma = kh$ can be treated as nonlinearity and dispersive parameters respectively. The Boussinesq theory accounts for the effects of nonlinearity δ and dispersion γ^2 to the leading order. However, when $\delta >> \gamma^2$, they reduce to Airy's equations which are valid for all δ; when $\delta << \gamma^2$ they reduce to the linearized approximation with weak dispersion. When $\delta \to 0$ and $\gamma^2 \to 0$, the classical linearized theory can be recovered. Extensive work on the shallow water waves and the usefulness of the characteristic method has been reported by Rahman [7] in his excellent book on water waves published by Oxford University Press (see also Mei [5]).

Normalized Boussinesq equations (see Rahman [8]) for one-dimensional waves can be written as

$$\frac{\partial \eta}{\partial t} + \delta \frac{\partial}{\partial x}\left(\eta \frac{\partial \phi}{\partial x}\right) + \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{31}$$

$$\frac{\partial \phi}{\partial t} + \eta + \frac{\delta}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 = \frac{\gamma^2}{3}\frac{\partial^3 \phi}{\partial x^2 \partial t} \tag{32}$$

Eliminating η from (31) and (32) we get a single equation in ϕ.

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = \frac{\gamma^2}{3}\frac{\partial^4 \phi}{\partial^2 x \partial^2 t} - \delta \frac{\partial}{\partial t}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2\right] \tag{33}$$

It is this partial differential equation which produces two important analytical solutions of physical interest as mentioned above. The mathematical expression for a solitary wave in nondimensional form can be written as

$$\eta = \operatorname{sech}^2\left\{\frac{(3\delta)^{\frac{1}{2}}}{2\gamma}(\xi - \xi_0)\right\} \tag{34}$$

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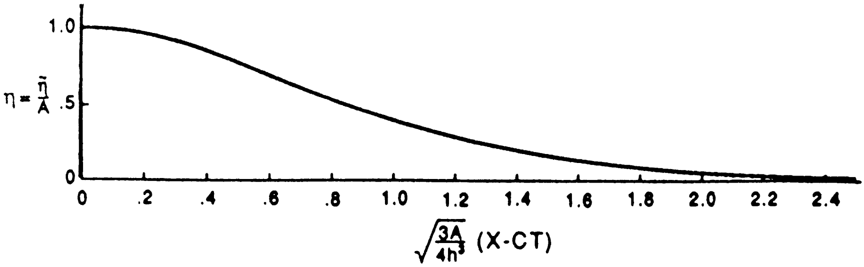

Figure 2: Dimensionless free surface profile of a solitary wave

where ξ_0 is an integration constant and can be chosen to be zero. In dimensional form the surface profile can be written as

$$\tilde{\eta} = A\mathrm{sech}^2\left\{\frac{\sqrt{3}}{2}\left(\frac{A}{h^3}\right)^{\frac{1}{2}}(X - CT)\right\} \tag{35}$$

This is called the solitary wave of Boussinesq [2]. The solitary wave form is shown in Figure 2. The A therefore represents the height of the wave and h the depth at infinity.

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