Long axisymmetrical waves in a viscous ocean around a circular island

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Abstract

The problem of long waves due to any applied normal surface stress in a viscous liquid ocean surrounding a circular island is solved. The expression of the surface displacement \( \zeta \) correct to \( O(\nu) \) terms, \( \nu \) being the coefficient of viscosity, is obtained. Assuming the stress to be time-periodic the surface displacement \( \zeta \), is shown to attain a steady-state and the resulting wave forms at any distance from the island are determined. Numerical studies of the results along with illustrations are provided to bring out the effect of viscosity and the presence of an island on the ocean waves otherwise devoid of them.

Keywords: coefficient of viscosity, long wave motion, asymptotic analysis, turbulence.

1 Introduction

The three-dimensional problem of short waves due to arbitrary initial time-dependent surface pressure together with an elevation of the surface in a viscous fluid of finite depth \( h \) without any other boundaries has been investigated by Nikitin and Potetyunko [2] and Bandyopadhyay [3, 4]. Assuming the same fluid regime, Oborotov studies the problem of free long waves in the axisymmetric case in which however the condition of zero tangential stress on the surface was not fully utilized. In the oceanographic context it is, however, natural to investigate problems in the presence of a boundary, enclosed or otherwise, in the ocean. Recently, Das and Ghosh analyzed the initial value problem of generation of storm surges by a symmetrically distributed and time-periodic surface wind near a circular island with variable topography [11,12]. For this purpose, Das used a depth-averaged Reynolds’ equation together with the usual assumption of shallow water wave theory and the f-plane approximation. It seems that hitherto
there has been no attempt to analyze the basic non-turbulent non-rotating problem of long axisymmetrical waves formed around a circular island due to suddenly applied arbitrary surface pressure in an ocean of viscous fluid of finite depth. Our objective is to study here this problem, taking into account the complete surface boundary condition of zero tangential stress (contrast Oborotov, who however does not consider any boundary). It is shown that the formal solution [equation (12) or (14)] is achieved by the application of Laplace and Weber-Orr transform. An apparently formidable expression of $\zeta$ [equation (16)] is next approximated for small coefficient of viscosity $\nu$ correct up to $O(\nu)$ terms under certain conditions on the applied pressure [see condition (A) in Section 3].

We next suppose that the applied pressure is periodic in time and deduce that the motion attains a steady-state as $t \to \infty$. The steady-state surface displacement $\zeta^*_s$ [equation (27)] is next analyzed to demonstrate that the surface displacement everywhere consists of a pair of oppositely moving progressive waves accompanied by standing local disturbances. The asymptotic character of these waves’ at large distances from the island is easily derived to give a simplified picture of the wave motion.

A numerical study is also undertaken to estimate the range of distances over which the exact solution can be approximated by its asymptotic counterpart and an illustration is also provided to study the nature of the waves to a particular pressure distribution with and without the presence of an island. Last but not the least we note that the Stokes-Navier equations become identical with Reynolds equation for turbulent motion when the horizontal and vertical exchange coefficients become equal, their common value playing the part of $\nu$ [13]. Consequently the solutions of $v_r, v_z$ and $\zeta$ [equations (10)–(12)] for the long wave axisymmetrical problem obtained here also give the solution of the mean fields in corresponding problem for turbulent motion under preceding condition and the various other conditions (e.g. no surface shear) assumed here.

2 The problem and solution

We consider an initially undisturbed ocean of a viscous homogeneous liquid of uniform depth $h$ and density $\rho$ surrounding a circular island of radius unity. Let $(r, \theta, z)$ denote the cylindrical coordinates of any point in the ocean with the undisturbed free surface taken as the xy-plane, the centre of the island as the origin O, and the vertically upward direction as the z-axis. Motion is generated at $t = 0^+$ by the action of an axisymmetrical surface pressure distribution $p_0(r, t)$.

Let $v_r, v_z$, and $\zeta$ denote respectively the velocities in the direction of increasing $r$ and $z$ and the surface displacement at any point and time.

The linearised Stokes-Navier equations of motion are

$$\frac{\partial v_r}{\partial t} = -\frac{1}{\rho} \frac{\partial p_1}{\partial r} + \nu(\nabla^2 v_r - \frac{v_r}{r^2}),$$

(1)
\[
\frac{\partial v_z}{\partial t} = -\frac{1}{\rho} \frac{\partial p_1}{\partial z} + v \nabla^2 v_z \tag{2}
\]

where

\[ p_1 = p + \rho g z, \quad \nabla^2 = \frac{\partial^2}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \]

The equation of continuity is

\[
\frac{\partial v_r}{\partial t} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0 \tag{3}
\]

The boundary and initial conditions from the continuity of tangential and normal stresses, the kinematic surface condition, the zero-velocity condition on a rigid surface, and the all-quiet condition of the fluid just before the start:

(i) on \( z = \zeta \approx 0, r > 1, \)
\[ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} = 0, \quad -p_1 + \rho g v_z + 2\mu \frac{\partial v_z}{\partial z} = -p_0, \quad \frac{\partial \zeta}{\partial t} = v_z (r, 0, t) \]

(ii) on \( z = -h, r > 1, v_r = v_z = 0; \) on \( r = 1, v_r = v_z = 0 \) (4)–(9)

(iii) at \( t = 0, v_r = 0 = v_z, \zeta = 0, \dot{v}_r = -\rho^{-1}(\partial p_0 / \partial r) \)

To the above equations and conditions, we apply the usual assumptions adopted for the long wave motion. These are that

(a) \( p \) is given by hydrostatic formula, and
(b) the vertical acceleration is negligible and the viscous force in the \( z \)-direction is negligible.

The formal solution of the problem may be affected by introducing the Laplace transform and the Weber-Orr transform (Erdélyi et al. [5])

\[
v_r = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} ds \int \frac{m^2 \tilde{p}_1}{D_1} \left[ (k^2 + m^2)(\cosh mz - \cosh mh) - k^2 \{\cosh m(z + h) - m h \sinh m(z + h) - 1\} \times [J_1(\nu r)Y_1(\nu r) - Y_1(\nu r)J_1(\nu r)] \right] kdk \tag{10}
\]

\[
v_z = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} ds \int \frac{\tilde{p}_1}{D_1} \left[ (k^2 + m^2)(\sinh mz + \sinh mh) + k^2 \nu h \cosh m(z + h) - \sinh m(z + h) - mh\right] - m(z + h) [(k^2 + m^2) \cosh mh - k^2]\times [J_0(\nu r)Y_1(\nu r) - Y_0(\nu r)J_1(\nu r)] + \frac{1}{kr}] k^2 dk \tag{11}
\]

\[
\zeta = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} ds \int \frac{\tilde{p}_0}{D_1} m^2 \left[ \sinh mh - mh \cosh mh \right] \times [J_0(\nu r)Y_1(\nu r) - Y_0(\nu r)J_1(\nu r)] + \frac{1}{kr}] k^2 dk \tag{12}
\]

where

\[ D_1 = m^2 [\nu s m k^2 (\cosh mh - 1) - g k^2 (\sinh mh - mh \cosh mh) + m^3 \nu \cosh mh], \]

\[ m^2 = k^2 + s / \nu . \]

In deriving the result for \( \zeta \) use has been made of the condition \( \zeta|_{t=0} = 0 \) and a Tauberian theorem (Sneddon, p.185, [7]).
From (12) one gets

\[
\zeta = \frac{1}{2\pi i \rho} \int_{c-i\infty}^{c+i\infty} \frac{(\mu h - \tanh \mu h)}{\cosh^2 (1 - \sec h \mu h) + v^2 (\tanh \mu h - \mu h)} \, e^{st} \bar{P}_0 \, ds \times \left[ J_0(\mu h)Y_1(\mu h) - Y_0(\mu h)J_1(\mu h) + \frac{1}{\mu h} \right] k^2 \, dk. \tag{13}
\]

By the inversion theorem of Laplace and the convolution theorem we get

\[
\zeta = \frac{1}{2\pi i \rho} \int_{c-i\infty}^{c+i\infty} \frac{F(v^{1/2})}{f(v^{1/2})} \, ds \left[ J_0(\mu h)Y_1(\mu h) - Y_0(\mu h)J_1(\mu h) + \frac{1}{\mu h} \right] k^2 \, dk
\]

where

\[
- F(v^{1/2}) = v^{1/2} \tanh \left( (s + \mu \nu^2) / \nu \right) ^{1/2} h - (s + \mu \nu^2)^{1/2} h
\]

\[
f(v^{1/2}) = v^{1/2} (s + \mu \nu^2) k^2 \left[ 1 - \sec h \left( \frac{s + \mu \nu^2}{\nu} \right) h \right] + s(s + \mu \nu^2)^{3/2} - \nu^2 \left[ v^{1/2} \tanh \left( \frac{s + \mu \nu^2}{\nu} \right) h - \sqrt{s + \mu \nu^2} h \right]
\]

it may be of interest to note that the modification of the surface condition (4) as done Oborotov [1] for the case of free waves in the absence of any initial applied surface pressure lead us to a solution of \( v_x, v_z \) and \( \zeta \) which are quite different from that of us obtained in (10) – (12) because of the presence of the island.

3 Wave motion in a slightly viscous fluid

We first state the following result for small \( \nu \)

\[
\zeta = \frac{1}{2\nu i \rho} \int_{0}^{\infty} \int_{c-i\infty}^{c+i\infty} \left[ \bar{P}_0(k, t - \tau) d\tau \right] e^{st} \left( \frac{T_i}{\Delta_i} \right) ds \times \left[ J_0(\mu h)Y_1(\mu h) - Y_0(\mu h)J_1(\mu h) + \frac{1}{\mu h} \right] k^2 \, dk + O(\nu^{3/2}) \quad \text{as} \quad \nu \to 0 \tag{16}
\]

provided \( \bar{P}_0 = O(k^{-3-n}) \), \( n' > 0 \), uniformly in \( \tau \), as \( k \to \infty \) (Condition A). Here

\[
T_i = hs - (\nu s)^{1/2} + \frac{1}{2} \nu k^2 h
\]

\[
\Delta_i = s^3 + \frac{1}{2} \nu k^2 (5s^2 + gk^2 h) - gk^2(\nu s)^{1/2} + gk^2 hs
\]

Evidently the s-integration of (16) in question has a branch point at \( s = 0 \) as well as possible poles arising from those zeros of \( \Delta_i \) for which \( T_i \neq 0 \). If \( \nu = 0 \),
\[ \Delta_i \] has simple zeros at \( s = 0 \) and at \( s = \pm i\sqrt{gh}k \). Corresponding to these three values of \( s \), namely \( s_{00} = 0 \) \( s_{10} = i\sigma \) \( s_{20} = -i\sigma \), where \( \sigma = \sqrt{gh}k \), we may assume the following expansion for the possible zeros \( s_j, j = 0, 1, 2 \) of \( \Delta_i \) in powers of \( v^{1/2} \):

\[ s_j = s_{j0} + s_{j1}v^{1/2} + s_{j2}v + s_{j3}v^{3/2} + \ldots \quad j = 0, 1, 2 \tag{17} \]

It is found that \( s_{01} = 0 \) and to our degree of accuracy, \( T_i(s_0) = T_i(vs_{02}) = 0 \). Leaving aside then the zero \( s_0 \) of \( \Delta_i \), and determining the other coefficients \( s_{j\ell} \) for \( \ell = 1, 2, 3 \), we find that the s-integration has two simple poles at \( s = s_j \approx \alpha_j \) (to our degree of accuracy) given as follows:

\[ \alpha_j = (-1)^{j-1}i\sigma - \left\{ 1 + (-1)^{j-1}i \right\} \frac{1}{2h} \left( \frac{\sigma v}{2} \right)^{1/2} - \left( k^2 + \frac{1}{4h^2} \right)v \tag{18} \]

The s-integration is now completed by using a modified Bromwich contour (Tranter, 1951, [8]), the result is

\[ \zeta = \frac{1}{\rho} \int_{\rho}^{\infty} \sum_{j=1}^{2} \frac{T_j(\alpha_j, k)}{\Delta_i(\alpha_j, k)} e^{\alpha_j t} - \frac{1}{2\pi i} \int_{0}^{\infty} e^{-\chi} \left[ \frac{T_1}{\Delta_1} \right]_{s=\chi e^{+i\pi}} - \left[ \frac{T_1}{\Delta_1} \right]_{s=\chi e^{-i\pi}} \right) d\chi \tag{19} \]

where \( f(k, t) = \sum_{j=1}^{2} \frac{T_j(\alpha_j, k)}{\Delta_i(\alpha_j, k)} e^{\alpha_j t} \)

\[ -\frac{1}{\rho} \int_{\rho}^{\infty} \sum_{j=1}^{2} \frac{T_j(\alpha_j, k)}{\Delta_i(\alpha_j, k)} e^{\alpha_j t} \left[ \sigma - \frac{\sigma_1}{2} \right] \sin(\sigma - \sigma_1)t + \left( \frac{\sigma_1}{2} + vk^2 \right) \cos(\sigma - \sigma_1)t \tag{20} \]

where

\[ \sigma_1 = \frac{1}{2h} \left( \frac{\sigma v}{2} \right)^{1/2}, \quad \sigma_2 = \sigma_1 + \left( k^2 + \frac{1}{4h^2} \right)v \tag{21} \]

\[ \zeta \]

The final expression for \( \zeta \) results when \( f(k, t) \), in (35) is modified in accordance with (21) and (22).

4 Surface displacement for an oscillatory pressure distribution and its asymptotic value for large distances in steady-state

When the applied pressure distribution is of the form \( p_0(x, t) = p_0(x)e^{i\omega t}H(t) \), (19) gives by (21) and (22)
\[ \zeta = \zeta_1 + \zeta_2 \]  

where

\[ \zeta_1 = \frac{1}{\rho g} \int_1^\infty -\sum_{j=1}^{2} (\alpha_j + iB_j) e^{i(\sigma - \sigma_1 - \omega - \sigma_2)} + A + iB \times F(k, r) \tilde{P}_0(k) e^{i\omega t} \, dk \]  

and

\[ \zeta_2 = -\frac{1}{2\pi i \rho} \int_1^\infty F(k, r) \tilde{P}_0(k) k^2 \, dk \times e^{i\omega t} \int_0^\infty e^{-(\sigma + \chi)\tau} \, d\tau \times \int_0^\infty \left\{ \left[ \frac{T_1}{\Delta_1} \right]_{\chi=\tau^{+}} - \left[ \frac{T_1}{\Delta_1} \right]_{\chi=\tau^{-}} \right\} d\chi \]  

where

\[ F(k, r) = \left[ J_0(\sigma r) Y_1(k) - J_0(\sigma r) Y_1(k) + 1 \right] \]

\[ A = \sigma^2 - \frac{3\sigma \sigma_1^2}{2} + \sigma_1^2, \quad B = \frac{(\sigma_1 + vk^2)\omega}{2}, \quad C = \sigma \left( \frac{\sigma_1}{2} + vk^2 - \sigma_2 \right), \quad D = \omega \left( \frac{\sigma_1}{2} - \sigma \right) \]

and \[ \alpha_j = \frac{A + (-1)^j D}{2}, \quad \beta_j = \frac{B + (-1)^j C}{2} \]

4.1 Exact form of the steady-state wave integral

We may write

\[ \zeta_2 = -\frac{1}{2\pi i \rho} \int_1^\infty F(k, r) \tilde{P}_0(k) k^2 \, dk \times e^{i\omega t} \int_0^\infty e^{-(\sigma + \chi)\tau} \, d\tau \times \int_0^\infty \left\{ \left[ \frac{T_1}{\Delta_1} \right]_{\chi=\tau^{+}} - \left[ \frac{T_1}{\Delta_1} \right]_{\chi=\tau^{-}} \right\} d\chi \]

It is easy that \( t \times \chi \) integral tends to zero as \( t \to \infty \) for \( 0 < \varepsilon < \tau' \leq 1 \), while the continuity of the \( \chi - \tau' \) integral in \( 0 \leq \varepsilon < 1 \) ensures that \( \zeta_2 \to 0 \) as \( t \to \infty \). Therefore in this case as \( t \to \infty \), \( r \) remaining fixed, the steady-state surface displacement \( \zeta^*_1 \) thus comes out to be \( \zeta^*_1 \) and it is given by

\[ \zeta^*_1 = -\frac{1}{\rho g} \int_1^\infty \frac{(A+iB)}{(\sigma_2+i\omega)^2 + (\sigma - \sigma_1)^2} F(k, r) \tilde{P}_0(k) e^{i\omega t} \, dk \]  

The dominant contribution of the wave integral corresponding to \( \zeta^*_1 \) comes from the poles of the integrand i.e. the points where

\[ R(k) \equiv (\sigma_2+i\omega)^2 + (\sigma - \sigma_1)^2 = 0. \]
For \( \nu = 0 \), \( R(k) \) has always two distinct real roots at
\( k = \pm \frac{\omega}{\sqrt{gh}} = (\beta_1, \beta_2) \) (say), of these \( \beta_2 \) is inadmissible since \( k > 1 \) and the zero
\( k_1 \) of \( R(k) \) corresponding to \( \beta_1 \) can be obtained from \( \beta_1 \) by iteration with respect
to \( \nu \) as
\[
k_1 = \beta_1 + \nu^{1/2}k_{11} + \nu k_{12}
\]
(29)
\( k_{ij} \) are determined by substituting (29) in (28) and equating the coefficients of
\( \nu^{1/2} \) and \( \nu \) of \( R(k) = 0 \) we get
\[
k_1 = \frac{\omega}{\sqrt{gh}} + \left( \frac{\omega}{4gh^3} \right)^{1/2} e^{-i\nu/4} \nu^{1/2} - i \frac{1}{\sqrt{gh}} \left[ \frac{3}{8h^2} + \frac{\omega^2}{gh} \right] \nu.
\]

To evaluate the k-integral of \( \zeta_1 \) we consider a contour which essentially is the
lower quadrant of a circle of radius \( R \) bounded by the lines \( x = 1 \) to \( x = R \) and \( y = 0 \) to \( y = -R \) and write the equation (27) as
\[
\zeta_1^* = -\frac{e^{i\nu t}}{\rho g} \left[ N_1(k_1) \int_i^\infty \frac{F(k,r)}{k-k_1} dk + \int_i^\infty \frac{N_1(k) - N_1(k_1)}{k-k_1} F(k,r) dk \right]
\]
(30)
where
\[
N_1(k) = \frac{(A + iB)'}{R'(k,v)},
\]
and ‘’ represents \( \frac{d}{dk} \). Since \( J_0(kr)Y_1(k) - Y_0(kr)J_1(k) = \text{Im} H_0^{(2)}(kr)H_1^{(1)}(k) \)
(cf. Erdélyi et al. [5]) the first term at the right-hand side of (30) may be written as
\[
\zeta_1^* = -\frac{e^{i\nu t}}{\rho g} \left[ N_1(k_1) \int_i^\infty \frac{\text{Im}[H_0^{(2)}(kr)H_1^{(1)}(k)] + 1/kr}{k-k_1} dk \right],
\]
then from (30), the exact value of the steady-state surface displacement is
\[
\zeta_1^* = -\frac{e^{i\nu t}}{\rho g} \left[ \pi \left\{ H_0^{(2)}(k_1 r)H_1^{(1)}(k_1) - \overline{H_0^{(2)}(k_1 r)H_1^{(1)}(k_1)} \right\} N_1(k_1) + \right.
\]
\[
+ \left. \int_i^\infty \frac{N_1(k) - N_1(k_1)}{k-k_1} F(k,r) dk - \frac{1}{k_1 r} \log(1 - k_1) \right],
\]
(31)
where \( H_1^{(1)}(z) \) and \( H_0^{(2)}(z) \) are Hankel functions of the first and second kind of
order one and zero respectively. It can be easily verified that the integral-free
term in the R.H.S. of (31) represents two progressive wave components in the
steady-state motion for all \( r \) and \( t \) with complex velocities \( \omega/k_1 \) and \( \omega/\overline{k_1} \). The
second term of (31) on the other hand represents in general a standing wave
having the same frequency as that of the applied disturbances.

Since \( H_0^{(2)}(kr) \sim \sqrt{2/\pi kr} e^{-i(kr-\pi/4)} \), and also since at large distances the
standing waves fade away in comparison to the leading term, denoted by \( \zeta_1^{**} \) of
the asymptotic expression of \( \zeta_1^* \), we may write, for \( r >>1 \)
The steady-state wave motion at large distances from the island thus consists of two damped simple harmonic progressive waves, having the same velocity of propagation. One of this recedes away from the island and the other moving towards it, both having the same exponential attenuation factor.

5 Illustrative cases

We assume a wind stress model to the one proposed by Jelesnianski [9] as cited by Johns et al [10]. Following this model we write

\[ P_0(r) = \tau_0 \left( \frac{r}{R} \right)^3 \text{ when } 0 < r < R \text{ and } P_0(r) = \tau_0 \left( \frac{R}{r} \right) \text{ when } r \geq R , \]

where \( R \) is the ratio of the radii of the maximum wind and the island. Then

\[
\tilde{P}_0(K) = \left[ \frac{\tau_0}{k^2 R} \left( 4 S_{3,0}(kR) J_1(kR) Y_1(k) - Y_1(kR) J_1(k) \right) - S_{4,1}(kR) \left[ J_0(kR) Y_1(k) - Y_0(kR) J_1(k) \right] \right] \\
+ \tau_0 R^2 S_{0,1}(kR) \left[ J_0(kR) Y_1(k) - Y_0(kR) J_1(k) \right] / \left[ J_1^2(k) + Y_1^2(k) \right]
\]

where \( S_{\mu,\nu}(z) \) denotes Lommel’s function.

Figure 1: Exact form of the progressive wave part.

Figure 2: Steady-state large distance wave form.
By means of this result and the values of the non-dimensional parameters \( \omega' = \omega \sqrt{h/g} \), \( \nu' = \nu \sqrt{gh^3} \) and \( k' = k h \), we obtain the corresponding illustrations of the waves from their exact as well as from asymptotic expression.

\[
\begin{align*}
\zeta_r & \quad \text{Figure 3: One illustrative comparison of the exact form of the progressive wave part with its asymptotic form.} \\
\zeta_r & \quad \text{Figure 4: Steady-state waves with and without the presence of an island. (31) and (32) respectively over different range of distances. A comparison of the nature of these steady-state waves is shown and also a comparison is shown with the large distance steady-state waves for the case of with and without the presence of an island. The Numerical work and the graphical illustrations are done with the help of ‘Mathematica’ software version 5.2.} \\
\zeta_r & \quad \text{6 Conclusion} \\
\zeta_r & \quad \text{Wave motion at high Reynolds number will certainly lead to the formation of boundary layers behind the leading wave around the island and also both at the surface and bottom of the open ocean. We will study the order of thickness of}
\end{align*}
\]
this layer in a subsequent study of this problem. Secondly, a through investigation of the corresponding asymmetrical problem along with its analytical solution will perhaps be a formidable task ahead of us. Lastly, a practical value of this problem may be related to the generation of long waves due to atmospheric point blast in the line with the centre of the island although in that case, I suppose, the non-linear effect of the motion should have to be incorporated.

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References