An exact solution of Navier-Stokes equations for the flow through a diverging artery

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Abstract

The estimation of the axial and radial component distributions of the blood velocity during its flow in a diverging artery is attempted. To this scope part of the Navier-Stokes equations and the continuity equation were treated analytically. Several flow characteristics, such as the back flow near the internal artery wall surfaces are investigated. Analytical results obtained by the proposed methodology are also included.

1 Introduction

Blood flow analysis through diverging arteries is very important due to the fact that the diverging arteries cause damage in the cardiovascular system. Although main physical mechanism of the formation of the arterial diverging is not clearly understood by the theoretical point of view, however, the effects of blood flow in the damage accumulation of the diverging arteries are serious.

A number of interesting studies [1-4] pertaining to blood flow through arteries have been carried out in order to investigate the blood flow characteristics in diverging or stenosed arteries. Several of these studies were based upon the assumption that a Newtonian fluid and a diverging or stenotic geometry can simulate the blood by a harmonic function. It is found that the above assumptions are reasonable especially for arteries above 1 mm in diameter.

An attempt is made here to evaluate the velocity profiles in a diverging artery, by treating the artery as a uniform thick cylindrical tube of isotropic and incompressible material, containing a Newtonian fluid. In the present analysis, an expression for the axial and radial velocity components distributions, which

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are expressed in terms of Bessel functions, is derived. The accuracy of the velocity estimation has been achieved by taking all the terms of Navier-Stokes equations.

A quantitative analysis is performed through computations of the important quantities presented graphically, followed by a discussion concerning the validity of the present results.

2 Formulation of the problem

A part of a diverging artery under consideration is modeled as an axisymmetric tube whose wall material is being simulated as an isotropic and incompressible material containing a Newtonian fluid representing the blood. Let (r, z) be the polar coordinates where the z-axis lie along the axis of symmetry of the artery where r is the radial direction. The geometry of the diverging segment is described as

$$R(z) = R_0 e^z \tag{1}$$

where R_0 is the constant radius of the normal artery in a non-diverging region, while z is the axial distance initiated from the end of the normal artery.

Let us consider the blood flow through the diverging segment to be steady, axisymmetric , two-dimensional and fully developed where the blood is treated to be incompressible Newtonian fluid. The Navier-Stokes equations and the equation of continuity that govern the motion of blood, in the cylindrical polar coordinates system can be written as:

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{r}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = 0 \tag{2}$$

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{1}{r}\frac{\partial p}{\partial r} + \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) = 0 \tag{3}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

where u and w are the radial and axial velocity components respectively, μ, ρ are the blood viscosity and density respectively and p is the pressure.

The above Navier-Stokes equations have been written by dropping the terms $\frac{\partial^2 w}{\partial z^2}$ and $\frac{\partial^2 u}{\partial z^2}$ from the right-hand terms since in several simulations it was verified that these terms are negligible in comparison with the radial derivatives.

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Assuming that the segment is axisymmetric and the axis of symmetry is straight, the analysis is restricted within the short part of the diverging artery.

Introducing a radial coordinate transformation given by $x = \dfrac{r}{R(z)}$ the equations of motion (2) – (4) with the boundary conditions take the following form:

$$\frac{1}{R}\left[xw\frac{dR}{dz} - u\right]\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} + \frac{\mu}{\rho R^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{x}\frac{\partial w}{\partial x}\right) - \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \qquad (5)$$

$$\frac{1}{R}\left[xw\frac{dR}{dz} - u\right]\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z} + \frac{\mu}{\rho R^2}\left(\frac{1}{x}\frac{\partial u}{\partial x} - \frac{u}{x^2}\right) - \frac{1}{R}\frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad (6)$$

$$\frac{1}{R}\frac{\partial u}{\partial x} + \frac{u}{xR} + \frac{\partial w}{\partial z} - \frac{x}{R}\frac{dR}{dz}\frac{\partial w}{\partial x} = 0 \qquad (7)$$

with

$w(x,z) = 0$ on the internal wall surface (8)

$u(x,z) = 0$ along the axisymmetric axis (9)

Taking into account the following dimensionless parameters

$$w = U_0 \overline{w} \qquad (10)$$

$$u = U_0 \overline{u} \qquad (11)$$

$$z = R_0 \overline{z} \qquad (12)$$

$$p = \rho U_0^2 \overline{p} \qquad (13)$$

$$\mathrm{Re} = \frac{\rho U_0 R_0}{\mu} \qquad (14)$$

$$R(z) = R_0 e^{\overline{z}} \qquad (15)$$

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where U_0 is the mean value of the axial velocity of the normal artery , the Navier-Stokes equations and the continuity equation can be written in the following way :

$$\left[x\overline{w}\frac{dR}{d\overline{z}} - \overline{u}\right]\frac{\partial \overline{w}}{\partial x} - \overline{w}\frac{\partial \overline{w}}{\partial \overline{z}} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \overline{w}}{\partial x^2} + \frac{1}{x}\frac{\partial \overline{w}}{\partial x}\right)e^{-2\overline{z}} - \frac{\partial \overline{p}}{\partial \overline{z}} = 0 \qquad (16)$$

$$\left[x\overline{w}\frac{dR}{d\overline{z}} - \overline{u}\right]\frac{\partial \overline{u}}{\partial x} - \overline{w}\frac{\partial \overline{u}}{\partial \overline{z}} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{1}{x}\frac{\partial \overline{u}}{\partial x} - \frac{\overline{u}}{x^2}\right)e^{-2\overline{z}} - \frac{1}{\rho}\frac{\partial \overline{p}}{\partial x}e^{-\overline{z}} = 0 \qquad (17)$$

$$\frac{1}{\overline{R}}\frac{\partial \overline{u}}{\partial x} + \frac{\overline{u}}{x\overline{R}} + \frac{\partial \overline{w}}{\partial \overline{z}} - \frac{x}{\overline{R}}\frac{d\overline{R}}{d\overline{z}}\frac{\partial \overline{w}}{\partial x} = 0 \qquad (18)$$

The boundary conditions with respect to x at the wall are:

$$\overline{u}(x,\overline{z})\Big|_{x=1} = 0 \quad , \qquad (19a)$$

$$\overline{w}(x,\overline{z})\Big|_{x=1} = 0 \qquad (19b)$$

while the conditions on the axis of symmetry are:

$$\overline{u}(x,\overline{z})\Big|_{x=0} = 0 \quad , \qquad (20a)$$

$$\frac{\partial \overline{w}}{\partial x}\Big|_{x=0} = 0 \qquad (20b)$$

After some standard operations, the axial and radial velocities can be satisfactorily expressed:

$$w(\overline{x},\overline{z}) = J_0(\overline{x})e^{-2\overline{z}} \qquad (21)$$

$$u(\overline{x},\overline{z}) = \overline{x}J_0(\overline{x})e^{-\overline{z}} \qquad (22)$$

with

$$\overline{x} = 5.5x \qquad (23)$$

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The above results satisfied the continuity equation (18) as well as the Navier-Stokes equation (17) with the assumption

$$\bar{u} = x\bar{w}\frac{d\bar{R}}{d\bar{z}} \tag{23a}$$

while the quantity

$$\frac{1}{\mathrm{Re}}\left(\frac{1}{x}\frac{\partial \bar{u}}{\partial x} - \frac{\bar{u}}{x^2}\right) - \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}} \tag{23b}$$

can be assumed negligible for Reynolds numbers within the diverging region, reducing the term $\frac{\partial \bar{p}}{\partial x}$ to zero.

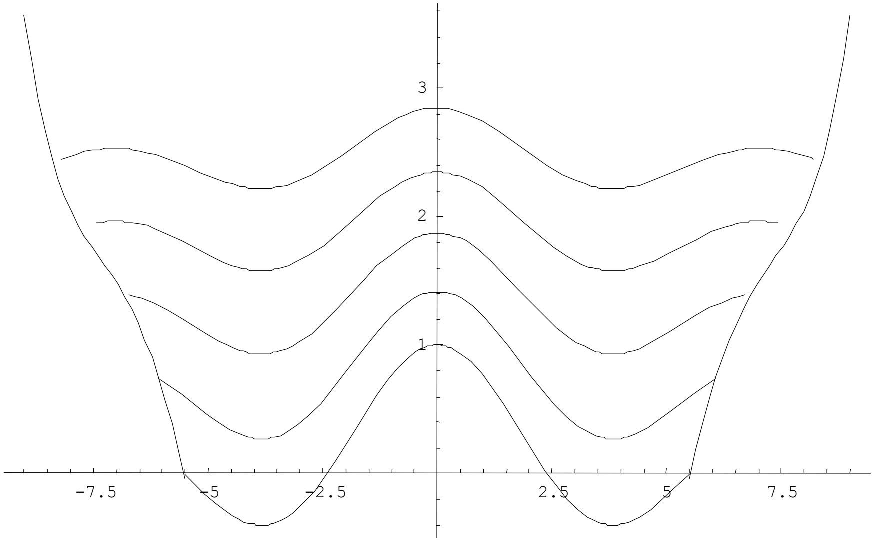

Figure 1: Axial velocity distribution along a diverging artery including the blood reversal near the wall surfaces.

3 Results and discussions

Considering the solutions (21) and (22) the graphical representation of the axial and the radial velocity components distribution within the region of the diverging segment is shown in figure 1. For calculations, the commercial code MATHEMATICA [5] has been used.

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The considered diverging segment has been modeled using the values R =1 mm, L=50 mm (length of the segment). The results of the axial velocity profiles of the blood flow are illustrated at the locations from z=0 to z=1 (see fig. 1).

The axial velocity curves are found to be decreasing as the radius r increasing while a reversal of flow begins near the wall, taking place during the diastolic segment. However, this backflow of the blood is eliminated for the region after the diastolic segment where the velocity profile is re-established.

The maximal forward velocity occurs in generally on the axisymmetric axis. Above results are in encourage agreement with measures obtained by other researchers [6-8] within the diverging regions(after arterial stenosis).

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