The linear and nonlinear properties of the high-order Boussinesq equations for wave propagation

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Abstract

A study of the Boussinesq equations for waves propagating from deep water to shallow water is presented. In this paper, we rederive the Boussinesq equations with the recursion form not only appearing in the main variables but in the coefficients. This greatly reduces the efforts of the derivation of the higher-order Boussinesq equations. Parameters concerning the linear and nonlinear wave properties are also derived to analyze the accuracy of the present models. The linear properties include the phase velocity, the group velocity and the particle velocities. The forcing terms of the continuity equation and the equation of motion are developed to analyze the nonlinear properties. By choosing a suitable water-depth parameter \( m \), the optimal wave models are consequently determined. Our model provides an easier and more flexible method to analyze the wave mechanics than previous studies based on the Padé approximation.

1 Introduction

In 1871, the classical Boussinesq equations first derived by Boussinesq [1] are frequently applied to analyze the wave mechanics in shallow water. As one century passed by, Peregrine [8] improved the classical Boussinesq equations for wave propagation over an uneven seabed. For most researchers, numbers of efforts they made are for the sake of extending the applicable depth range of Boussinesq equations wider. Consequently, various kinds of the optimal wave equations were evaluated in the following two decades. Witting [9] applied
conservative equations to analyze wave mechanics in a constant-depth channel. By applying the (2,2) Padé approximation, Witting obtained good results for both deep and shallow water waves.

In 1991, Madsen et al. [4] formulated the conventional Boussinesq equations for the flat bottom in terms of volume flux components instead of the depth-averaged velocity. One year later, Madsen and Sorensen [5] further extended this set of Boussinesq-type equations for a slowing varying topography and introduced the linear shoaling gradient as another quantity to evaluate the improvement of wave equations. Nwogu [7] derived a new set of modified Boussinesq equations in terms of a horizontal velocity at an arbitrary elevation. His study is a crucial beginning for other consequent studies. He makes the new set of equations applicable to wave propagation from relatively deep water to shallow water. Chen and Liu [2] rederived the $O(\mu^2)$ Boussinesq-type equations in terms of an arbitrary-depth velocity potential instead of the horizontal velocity adopted by Nwogu. Their investigation provides an optimal wave model at a specific elevation near the middle depth that is lightly different from Nwogu’s result.

In 2000, Gobbi et al. [3] derived the $O(\mu^4)$ Boussinesq equations by introducing a new variable defined as a weighted average of the velocity potential at two distinct water depths. They determined the values of two parameters in the weighted velocity potential and so obtained the best Boussinesq-type model by comparing the coefficients of the Taylor-like expansion of dispersion relation with those of Padé (4,4) approximation of the exact linear solution. In 2002, Madsen et al. [6] introduced finite series approximations involving up to fifth-derivative operators. By using the Padé approximants, they make some wave characteristics very accurate within the range of $\mu$ from 0 to 20.

Our present paper is organized as follows. The $O(\mu^4)$ Boussinesq equations in terms of a velocity potential at an arbitrary water depth are first derived in Section 2. After deriving the set of high-order Boussinesq equations, the linear wave properties including the phase velocity and the particle motions are compared to those of the exact solution and discussed in Section 3. In Section 4, two series of parameters obtained from the forcing terms of the continuity equation and the equation of motion are developed to analyze the nonlinear wave properties. To generate the optimal models, we choose a suitable value of $m$ which is yielded by taking the arithmetic mean of the optimal $m$ of all properties in Section 5. Some of linear and nonlinear characteristics are compared to those of the exact solution. Conclusions are made in Section 6.

2 Derivation of the $O(\mu^4)$ Boussinesq equations

The full boundary value problem for potential flow is given in terms of non-dimensional variables by
\[ \mu^2 \nabla^2 \Phi + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{at} \quad -h \leq z \leq \varepsilon \eta \]  

(1)

\[ \frac{\partial \Phi}{\partial z} = \mu^2 \left( \frac{\partial \eta}{\partial t} + \varepsilon \Phi \cdot \nabla \eta \right) \quad \text{at} \quad z = \varepsilon \eta \]  

(2)

\[ \frac{\partial \Phi}{\partial z} = -\mu^2 (\nabla \Phi \cdot \nabla h) \quad \text{at} \quad z = -h \]  

(3)

\[ \frac{\partial \Phi}{\partial t} + \eta + \frac{\varepsilon}{2} \left[ (\nabla \Phi)^2 + \frac{1}{\mu^2} \left( \frac{\partial \Phi}{\partial z} \right)^2 \right] = 0 \quad \text{at} \quad z = \varepsilon \eta \]  

(4)

Two parameters measuring nonlinearity and dispersive effects are denoted as \( \varepsilon \) \((\equiv a_0 h_0^{-1})\) and \( \mu \) \((\equiv k_0 h_0)\) respectively. The relation, \( O(\varepsilon) = O(\mu^2) < 1 \), is assumed to describe the weakly nonlinear wave propagation. Now we integrate Eq.(1) from \( z = -h \) to \( z = \varepsilon \eta \) and applying the boundary conditions Eq.(2) and Eq.(3) into it, it yields

\[ \nabla \cdot \int_{-h}^{\varepsilon \eta} \nabla \Phi dz + \frac{\partial \eta}{\partial t} = 0 \]  

(5)

The velocity potential \( \Phi \) is assumed as

\[ \Phi(x, y, z, t) = \sum_{0}^{\infty} \mu^{2n} \Phi_n (x, y, z, t) \]  

(6)

Substituting Eq.(6) into Eq.(1) and Eq.(3), the general solutions of \( \Phi_0 \), \( \Phi_1 \), and \( \Phi_2 \) can be expressed as

\[
\begin{align*}
\Phi_0 &= \Phi_{00} (x, y, t) \\
\Phi_1 &= \Phi_{10} (x, y, t) + z \Psi_1 - \frac{z^2}{2!} \nabla^2 \Phi_{00} \\
\Phi_2 &= \Phi_{20} (x, y, t) + z \Psi_2 - \frac{z^2}{2!} \nabla^2 \Phi_{10} - \frac{z^3}{3!} \nabla^2 \Psi_1 + \frac{z^4}{4!} \nabla^2 \nabla^2 \Phi_{00}
\end{align*}
\]  

(7)

where

\[
\begin{align*}
\Psi_1 (x, y, h, t) &= -\nabla \cdot (h \nabla \Phi_{00}) \\
\Psi_2 (x, y, h, t) &= -\nabla \cdot (h \nabla \Phi_{10}) + \frac{h^2}{2!} \nabla^2 \Psi_1 + \frac{h^3}{3!} \nabla^2 \nabla^2 \Phi_{00} + \left( h \nabla \Psi_1 + \frac{h^2}{2!} \nabla \nabla^2 \Phi_{00} \right)
\end{align*}
\]  

(8)

Observing the expressions of Eq.(7) and (8), the recursion forms appear not only in the terms \( \Phi_i \) but also in the terms \( \Psi_i \). The recursion forms greatly make the derivation of the high-order equations more efficient. Consequently, the
corresponding variables can be represented in the recursion forms. Now we define $\Phi_m$ which indicates the velocity potential at the elevation $z = zm$, it yields

$$\Phi_m = \Phi_0 + \mu^2 \left[ \Phi_{10} + zm \Psi_1 - \frac{z_m^2}{2!} \nabla^2 \Phi_0 \right]$$

$$+ \mu^4 \left[ \Phi_{20} + zm \Psi_2 - \frac{z_m^2}{2!} \nabla^2 \Phi_{10} - \frac{z_m^3}{3!} \nabla^2 \Psi_1 + \frac{z_m^4}{4!} \nabla^2 \nabla^2 \Phi_0 \right] + O(\mu^6)$$

(9)

From the above equation, we can obtain $\Phi_0, \Phi_{10}, \Phi_{20}$ respectively. By introducing Eq.(8), the velocity potential $\Phi$ can be rewritten as

$$\Phi = \Phi_m + \mu^2 \left[ (z - zm) \Psi_1 - \frac{(z^2 - z_m^2)}{2!} \nabla^2 \Phi_0 \right] + \mu^4 \left[ (z - zm) \Psi_2 \right.$$  

$$- \frac{(z^2 - z_m^2)}{2!} \nabla^2 \Phi_{10} - \frac{(z^3 - z_m^3)}{3!} \nabla^2 \Psi_1 + \frac{(z^4 - z_m^4)}{4!} \nabla^2 \nabla^2 \Phi_0 \right] + O(\mu^6)$$

(10)

Inserting Eq.(10) into Eq.(4) and Eq.(5), then the set of the $O(\mu^4)$ Boussinesq equations can be generated. To analyze the fully nonlinear Boussinesq equations, the constant water depth $h$ is considered in the following analysis. First, we introduce a new parameter $m$ to indicate the specific elevation $zm (= mh)$. Obviously, the value of $m$ is between $-1$ (for the bottom) and zero (for the free surface). In previous studies, another depth parameter $\alpha$, which is frequently used to denote the elevation, can be easily converted to the present water-depth parameter $m$. Now the variables appeared in Eq.(7) can be represented by $m$ as follows

$$\begin{align*}
\Phi_0 &= \Phi_m \\
\Phi_{10} &= \left( m + \frac{1}{2} m^2 \right) h^2 \nabla^2 \Phi_m \equiv H_1 h^2 \nabla^2 \Phi_m \\
\Phi_{20} &= \left( \frac{1}{3} m + m^2 + \frac{5}{6} m^3 + \frac{5}{24} m^4 \right) h^4 \nabla^2 \nabla^2 \Phi_m \equiv H_2 h^4 \nabla^2 \nabla^2 \Phi_m
\end{align*}$$

(11)

and

$$\begin{align*}
\Psi_1 &= -h \nabla^2 \Phi_m \equiv -G_1 h \nabla^2 \Phi_m \\
\Psi_2 &= -\left( \frac{1}{3} + m + \frac{1}{2} m^2 \right) h^3 \nabla^2 \nabla^2 \Phi_m \equiv -G_2 h^3 \nabla^2 \nabla^2 \Phi_m
\end{align*}$$

(12)
If nonlinear terms are neglected, the linear set of the \( \mathcal{O}(\mu^6) \) Boussinesq equations is as follows

\[
\frac{\partial \eta}{\partial t} + G_1 h \nabla^2 \Phi + \mu^2 \left( G_2 h^2 \nabla^2 \Phi \right) + \mu^4 \left( G_3 h^4 \nabla^2 \nabla^2 \Phi \right) + \mathcal{O}(\mu^6) = 0 \tag{13}
\]

\[
\eta + \frac{\partial \Phi}{\partial t} + \mu^2 \left( H_1 h^2 \nabla^2 \frac{\partial \Phi}{\partial t} \right) + \mu^4 \left( H_2 h^4 \nabla^2 \frac{\partial \Phi}{\partial t} \right) + \mathcal{O}(\mu^6) = 0 \tag{14}
\]

### 3 Linear analysis

Based on Eq.(13) and (14), five linear characteristics will be introduced and analyzed in this section. Combining Eq.(13) and (14), the dispersion relation can be obtained as

\[
C = \left[ \frac{1 - G_2 \mu^2 + G_3 \mu^4}{1 - H_1 \mu^2 + H_2 \mu^4} \right]^{0.5} \tag{15}
\]

The corresponding group velocity, \( C_g \), takes the form

\[
C_g = \frac{\partial}{\partial \mu} \left\{ \mu \left[ \frac{1 - G_2 \mu^2 + G_3 \mu^4}{1 - H_1 \mu^2 + H_2 \mu^4} \right]^{0.5} \right\} \tag{16}
\]

Next, three water particle properties will be investigated. Based on the assumption of the periodic solutions of Eq.(13) and (14), the parameters concerning the horizontal and vertical components of velocity of fluid particles, which are normalized by the values at the still water elevation, are represented as

\[
F_H = \frac{u(z,m)}{u(0,m)} = \left\{ 1 - \left[ (m-z)G_1 - \frac{z^2-m^2}{2} \right] \mu^2 \right. \\
\left. + \left[ (m-z)G_2 - \frac{(z^2-m^2)}{2} H_1 + \frac{(z^3-m^3)}{3} G_1 + \frac{(z^4-m^4)}{4} \right] \mu^4 \right\} \\
\left\{ 1 - \left[ mG_1 + \frac{m^2}{2} \right] \mu^2 + \left[ mG_2 + \frac{m^2}{2} H_1 - \frac{m^3}{3} G_1 - \frac{m^4}{4} \right] \mu^4 \right\} \tag{17}
\]

\[
F_V = \frac{w(z,m)}{w(0,m)} = \frac{(-G_1 - z) \mu^2 - \left( -G_2 - z H_1 + \frac{z^2}{2} G_1 + \frac{z^3}{3} \right) \mu^4}{(-G_1) \mu^2 - (-G_2) \mu^4} \tag{18}
\]
The parameter indicating the particle trajectory at the free surface is as

\[
F_T = \frac{w(0,m)}{u(0,m)} \cdot \frac{1}{\mu} \left[ (-G_1)\mu - (-G_2)\mu^3 \right]
\]

\[
1 - \left( mG_1 + \frac{m^2}{2!} \right) \mu^2 + \left( mG_2 + \frac{m^2}{2!}H_1 - \frac{m^3}{3!}G_1 - \frac{m^4}{4!} \right) \mu^4
\]

(19)

These five linear parameters shown in above equations will be compared to the exact solutions of the Stokes theory in order to decide the optimal wave models in the next section.

4 Nonlinear analysis

In this section, two series of nonlinear are obtained from the equation of continuity and the equation of motion respectively. We first introduce the perturbation expansion to the free surface elevation and the velocity potential as

\[
\Phi_m = \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + ...
\]

(20)

\[
\eta = \eta_0 + \epsilon \eta_1 + \epsilon^2 \eta_2 + ...
\]

(21)

By applying Eq.(20) and (21) into Eq.(5), we order the equation by powers of \(\epsilon\). At each order \(O(\epsilon^n)\), it yields (setting \(h = 1\))

\[
\frac{\partial \eta_n}{\partial t} + L_1 \Phi_n = -\nabla \cdot F_n
\]

(22)

where the operator \(L_1\) is

\[
L_1 = \left( G_1 \nabla^2 + \mu G_2 \nabla^2 \nabla^2 + \mu^4 G_3 \nabla^2 \nabla^2 \nabla^2 \right)
\]

(23)

Consequently, we obtain

\[
F_0 = 0
\]

(24.a)

\[
F_1 = \eta_0 \left( \nabla + \mu \nabla^2 \right) \Phi_0
\]

(24.b)

\[
F_2 = \left( \eta_0 \nabla \Phi_1 + \eta_1 \nabla \Phi_0 \right) + \mu^2 \left[ \left( mG_1 + \frac{m^2}{2!} \right) \eta_1 \nabla^2 \Phi_0 + \eta_0 \nabla^2 \Phi_1 \right]
\]

(24.c)
Similarly, we apply Eq.(20) and (21) into Eq.(4). At each order $O(\varepsilon^n)$, it also yields (setting $h = 1$)

\[ \eta_n + L_2 \frac{\partial \tilde{\Phi}_n}{\partial t} = -E_n \]  \hspace{1cm} (25)

where

\[ E_0 = 0 \] \hspace{1cm} (26.a)

\[ E_1 = -L_{2a} \frac{\partial \tilde{\Phi}_0}{\partial t} + \frac{1}{2}(\nabla L_2 \tilde{\Phi}_0)^2 + \mu^2 \left[ \frac{G_1^2}{2}(\nabla^2 \tilde{\Phi}_0)^2 \right] + \mu^4 \left[ G_1 G_2 \nabla^2 \tilde{\Phi}_0 \nabla^2 \tilde{\Phi}_0 \right] \]  \hspace{1cm} (26.b)

\[ E_2 = -L_{2a} \frac{\partial \tilde{\Phi}_1}{\partial t} - L_{2b} \frac{\partial \tilde{\Phi}_0}{\partial t} + \nabla L_2 \tilde{\Phi}_0 (\nabla L_2 \tilde{\Phi}_1 - \nabla L_{2a} \tilde{\Phi}_0) \]

\[ + \mu^2 \left[ G_1 \eta_0 (\nabla^2 \tilde{\Phi}_0)^2 \right] + G_1^2 \nabla^2 \tilde{\Phi}_0 \nabla^2 \tilde{\Phi}_1 + \mu^4 \left[ G_1 G_2 \nabla^2 \tilde{\Phi}_0 \nabla^2 \tilde{\Phi}_1 + \nabla^2 \tilde{\Phi}_1 \nabla^2 \tilde{\Phi}_0 \right] \]

\[ + (G_1 H_1 + G_2) \eta_0 \nabla^2 \tilde{\Phi}_0 \nabla^2 \tilde{\Phi}_0 \]  \hspace{1cm} (26.c)

with the operators

\[
\begin{align*}
L_2 &= 1 + \mu^2 H_1 \nabla^2 + \mu^4 H_2 \nabla^2 \\
L_{2a} &= \mu^2 G_1 \eta_0 \nabla^2 + \mu^4 G_2 \eta_0 \nabla^2 \\
L_{2b} &= \mu^2 \left( G_1 \eta_1 + \frac{\eta_0}{2} \right) \nabla^2 + \mu^4 \left( G_2 \eta_1 + \frac{H_1 \eta_0}{2} \right) \nabla^2 \nabla^2 
\end{align*}
\]  \hspace{1cm} (27)

Theoretically, all the parameters $F_n$ and $E_n$ can be analyzed. In present paper, the nonlinear parameters $F_1$ and $E_1$ will be investigated. In order to obtain the optimal Boussinesq equations, both the linear and the nonlinear parameters will be considered in next section to make the closest approach to those of the Stokes theory by choosing a suitable water-depth parameter $m$.

5 The optimal Boussinesq equations

The different kinds of Boussinesq-type equations can be formulated by choosing the water-depth parameter $m$. In shallow water with the condition $0 < \mu < 1$, all kinds of equations provide a excellent prediction of all properties for wave propagation. However, as waves travel in medium or deep water, the choice of $m$ will strongly dominate both the behaviors of nonlinear and linear characteristics of waves. Namely, though each linear or nonlinear characteristic has its own optimal value of $m$, we still need to determine the optimal $m$ to
balance all behaviors. Many studies based on the Padé approximation often face the dilemma that the most excellent behavior of dispersion relation always leads to poor predictions of other properties, for example, the particle velocities. In this section, we'll provide a new way to balance and overcome such a problem. In general cases, all nonlinear and linear properties should be considered in an equivalent status. Hence the value of $m$ will be generated by taking the arithmetic mean of the optimal $m$ of all properties. The corresponding optimal values of $m$ are -0.346 and -0.581 for the $O(\mu^2)$ and $O(\mu^4)$ models respectively. From now on, these three values will be adopted to examine all available properties of waves. Observing the phase velocities shown in Fig. 1, the $O(\mu^2)$ model represents a poor behavior that the curve diverges rapidly as $\mu \geq 3$. The $O(\mu^4)$ model shows the almost equal accuracy within the entire range. Figure 2 displays the horizontal and vertical velocities of water particles at several water depths. The depths we choose are $z = -0.5$. The $O(\mu^4)$ model gives the better approximation to the exact solution.

Figure 1: The phase velocity (solid line: the $O(\mu^4)$ model, dash line: the $O(\mu^2)$ model).

As for the nonlinear properties, two parameters nonlinear parameters, $E_i$ and $F_i$, which concern the equation of motion and the equation of continuity are slightly different from the transfer coefficients appeared in Gobbi et al. [3] and Madsen et al. [6]. The transfer coefficients are obtained by introducing the Stokes-like expansion of wave amplitude and the concept of the bound waves. Here we directly analyze the forcing terms of two nonlinear equations, Eq.(22) and Eq.(25), without making any assumption of the wave types. Figure 3
displays the phenomenon of the forcing terms \((E_1, F_1)\). The \(O(\mu^4)\) model still shows a better behavior than the other model.

Finally, we should emphasize again that the optimal values of \(m\) used in this section are obtained by taking the arithmetic mean of the optimal \(m\) of all linear and nonlinear properties.

Figure 2: The horizontal and vertical velocity of various models (solid line: the \(O(\mu^4)\) model, dash line: the \(O(\mu^2)\) model).

Figure 3: The nonlinear property \(E_1, F_1\) (solid line: the \(O(\mu^4)\) model, dash line: the \(O(\mu^2)\) model).

6 Conclusions

In the present study, the \(O(\mu^4)\) Boussinesq equations are derived in terms of the velocity potential at an arbitrary elevation. A recursion form is adopted to make the derivation more efficient. The corresponding linear and nonlinear parameters are also generated. By taking the arithmetic mean of the optimal water-depth parameter of \(m\) of all linear and nonlinear properties, the optimal Boussinesq equations are determined. Analyzing the optimal equations in the specific range of \(\mu = 0\) to \(\mu = 20\), the \(O(\mu^4)\) model doubtless expresses the best accuracy than the other low-order models. In conclusion, our model provides an easy and flexible method to analyze the wave mechanics than previous studies based on the Padé approximation.
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References


