Transcritical transitions in swirling quasi-columnar flows

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Abstract

Complete mathematical similarity between nonlinear transcritical transitions in swirled flows and transonic phenomena in compressible gas dynamics is established. It is shown that transition of base supercritical columnar swirling flow to another subcritical columnar swirling flow is governed by an equation identical to that of the shock waves formation in gases. It is proven that under some assumptions transition follows the scenario suggested by Benjamin [1]. The presented study is based on the nonlinear wave theory of swirled flows developed by Leibovich [7]. The results are established through a rigorous mathematical analysis and provide a solid theoretical understanding of the wave dynamics of axisymmetric swirling flows. The unsteady wave analysis suggests a consistent explanation of the mechanism leading to the axisymmetric vortex breakdown phenomenon in high-Reynolds swirling flows.

Keywords: vortex breakdown, shock wave, subcritical, supercritical, transonic, transcritical.

1 Introduction

Vortex breakdowns are organized structures that usually lead to turbulent flow. Most observers associate the term with a condition in which there is a rapid axial deceleration of the flow leading to a stagnation point and region of reversal axial flow. The complexities of this phenomenon are such that no generally accepted view of the underlying mechanisms responsible for it has emerged despite the importance of the problem and the efforts expended on it over a period of more than forty years. Several reviews (Hall [5], Leibovich [9],[10], Rusak and Wang [14], Lucca-Negro & O’Doherty [12]) outline the few theoretical developments which have been advanced by pioneers and classics working in this area. Further
progress in the vortex breakdown physics is being associated presently mainly with CFD modelling. However, there are basic issues in vortex breakdown physics that must be clarified on the theoretical level.

Our consideration is based on the results by Benjamin [1] and Leibovich [7]. According to Benjamin [1] the vortex breakdown is transition from supercritical approach flow to downstream ”conjugate” axisymmetric subcritical flow. The transition connecting the two flow states is outside the scope of the Benjamin’s theory. Leibovich [7] applied the multi-scale series expansion technique developed by Benney [4] to derive the nonlinear wave equation governing the inertia waves in columnar swirling flows. This equation is essentially the well-known Korteweg-de Vries equations and describes also waves propagating in shallow water. It admits steady solutions — solitons: the waves that preserve the permanent shape and speed. Leibovich [7] linked solitons with the formation of recirculation zones in vortex breakdown. However these solutions do not describe class of transitions considered by Benjamin. In our paper an attempt to describe the transition and to fill the gaps in Benjamin’s theory is presented. It is shown that under some assumptions the transition between two columnar flow states is governed by single Korteweg-de Vries-.Burgers equation, and at least the “weak” transitions follow the path suggested by Benjamin. This evolutional approach makes transparent the mechanism of formation of transition and provides deeper physical insight into the physics of transitions.

2 Flow model accounting for the transitions between different flow states with reference to Benjamin’s vortex breakdown theory

Consider high Reynolds axisymmetric viscous swirled incompressible flows governed by Navier-Stocks equations with constant viscosity $\nu$. Written in terms of non-dimensional dependent and independent variables

$$y = b^2 y', \quad z = L z' ,$$

$$\Psi = b^2 V_0 \Psi', \quad \Gamma = b V_0 \Gamma' ,$$

$$t = L/V_0 t'$$

where stream function $\Psi$ and circulation $\Gamma$ are defined in a common way

$$rv_r = \Psi_r, \quad rv_z = -\Psi_z, \quad \Gamma(r, z, t) = rv_\phi(r, z, t);$$

$v_r, v_z, \nu_\phi$ are radial, axial and azimuthal velocity components respectively;

$y = r^2; \quad L, b, V_0$ - characteristic axial, radial scales and velocity, and derivatives are represented by subscripts, the Navier-Stocks equations have the form (primes marking dimensionless variables are omitted below)
\[
D_{(k)} \frac{2}{2} \Psi_i + 2 \Psi_j D_{(k)} \frac{2}{2} \Psi_j + \frac{2}{2} \Gamma \Gamma_j - 2 \Psi \left[ y^{-1} D_{(k)} \frac{2}{2} \Psi \right] = \tilde{\mu} D_{(k)} \frac{2}{2} \Psi \\
(1)
\]

where \( \tilde{\mu} = \frac{V}{LV_0} \) is the inverse Reynold’s number based upon the wavelength.

\[
D_{(k)} = 4y \left( \frac{1}{y} \right) + k^2 \left( \frac{1}{z} \right) \quad \text{differential operator, } k = b / L.
\]

Flow eqns (1) for steady non-viscous columnar flow are equivalent to

\[
4y \Psi_{yy} = yH(\Psi) - \Gamma_y (\Psi) \Gamma_y (\Psi)
\]

\[
(2)
\]

\[
\Gamma = \Gamma(\Psi), \quad H = H(\Psi)
\]

where \( H(\Psi) \) - total head.

We now seek an axilll-symmetric long wave solutions of eqns (1) with small amplitude measured by the parameter \( \varepsilon \) propagating along the columnar swirled flow with stream function \( \Psi_0(y) \) and circulation \( \Gamma_0(y) \). Thus, we put

\[
\Psi = \Psi_0(y) + \varepsilon \psi(y, z, t)
\]

\( \Gamma = \Gamma_0(y) + \varepsilon \gamma(y, z, t) \) Assume now that the flow Reynolds number is large: \( \tilde{\mu} \ll 1 \). We include the viscous effects into the scheme of multi-scale asymptotic expansions considered by Leibovich [7] as follows

\[
\psi = \varepsilon \phi_0 A + \frac{1}{2} \varepsilon \phi_1 A^2 + k^2 \phi_2 A + \mu \phi_3 B + \mu \phi_4 A + \mu \phi_5 A_{\text{vis}}
\]

\[
(3)
\]

\[
\gamma = \gamma_0 A + \frac{1}{2} \varepsilon \gamma_1 A^2 + k^2 \gamma_2 A + \mu \gamma_3 B + \mu \gamma_4 A + \mu \gamma_5 A_{\text{vis}}
\]

\[
(4)
\]

\[
A_i = -c_0 A + \varepsilon c_1 A A + k^2 c_2 A + c_3 A + c_4 A + c_5 A_{\text{vis}}
\]

\[
(5)
\]

Three last terms in the expansions (3)-(5) are due to viscosity, others have been considered by Leibovich [7]. On substituting the expansions (3) - (5) into eqns (1), gathering terms containing \( \varepsilon, \varepsilon^2, \varepsilon^k, \mu \) as multipliers and equating the
expressions near these multipliers to zero we come to set of equations relative unknown functions $\phi_i, i = 0, 1, \ldots, 5$.

$$L\phi_i = f_i$$

with boundary conditions

$$\phi_i(0) = \phi_i(R) = 0$$

Functions $\phi_i, i = 0, 1, 2$ have been investigated by Leibovich [7]. In the equations for viscous functions $\phi_3, \phi_4, \phi_5$

$$f_3 = \frac{1}{4} \left( \frac{1}{W-c_0} \right) \left[ 4 (y\phi_0^+ - c_1\phi_0^-) - \frac{1}{4y^2} \left( \frac{\Gamma_0}{W-c_0} \right)^2 \left[ 4y \left( \frac{\phi_i \Gamma_{0y}}{W-c_0} \right) + c_3 \frac{\phi_4 \Gamma_{0y}}{W-c_0} \right] \right]$$

$$f_4 = -\frac{1}{y^2} \left( \frac{\Gamma_0 \Gamma_{0y}}{W-c_0} \right) \left( k^2 - c_4 \right) \phi_0 + \frac{1}{W-c_0} \left( 2k^2 - c_4 \right) \phi_0^+ - \frac{1}{4y} \phi_4$$

$$f_5 = \frac{1}{4y(W-c_0)} \left[ \frac{\phi_0}{4y} + c_4 \frac{1}{y(W-c_0)} \left[ \frac{1}{y} \left( \frac{\Gamma_{0y}}{W-c_0} \right)^2 + yq \right] \phi_0 \right]$$

$W$ - axial velocity of the base flow. Similar to non-viscous case the constants $c_1, c_4, c_5$ are determined due to compatibility conditions

$$\int_0^R f_i \phi_0 dr = 0$$

The first viscous term in the expansions is due to radial mixing, the third – due to axial mixing and the second is due to both axial and radial mixing. Both the functions and constants must be calculated with concrete base flow. We reduce our further analysis by the case when the latter viscous effect is dominant and the following equation for $A$ is valid

$$-c_0 A_z + \alpha c A_z + k^2 c_2 A_{zz} + c_4 A_{zz} = 0 \quad (6)$$

Leibovich [7] considered the wave equation without viscous term. He derived the well-known Korteweg-de Vries equation valid also for the waves propagating in a shallow water. The equation captures two important factors: nonlinearity and dispersion. Under some conditions the Korteweg-de Vries equation has steady solutions – solitons: waves that preserve permanent shape and speed.
\[ A = a \sec h^2 \left[ \frac{1}{2} \left( \frac{c_a}{3c_s^2} \right)^{\frac{1}{2}} \left( z - c_s t + \frac{1}{3} ac_s \epsilon \right) \right], \quad a = \text{const}. \]

For \( a = \frac{3c_o}{c_s \epsilon} \) these solution represents steady solution. Leibovich [7] connected this solution with the formation of recirculation zone in vortex breakdown. Remind that Benjamin’s [1] theory of vortex breakdown leaves open the important question of the structure of the transition region joining two conjugate flow states. For the solitary wave solutions the flow states far upstream and downstream are identical. Consequently, both the soliton solutions and Korteveg-de Vries equation cannot be applied to describe transitions between different flow states discussed by Benjamin [1]. From the other side the correct nonlinear wave theory must be able to cover this class of transitions. Kirbus and Leibovich [11] have confirmed the existence of Benjamin’s like transitions by numerical simulations.

The appearance in the wave eqn (6) of the second derivative over \( z \) results in the dramatic consequences. The wave equation admits shock wave-like solutions that meet boundary conditions

\[ A \to 0, A_z \to 0 \text{ if } z \to -\infty, A_z \to 0 \text{ if } z \to \infty, \]

and have different asymptotics far upstream and downstream. Integrating (6) from \(-\infty\) to \(+\infty\) due to boundary conditions we get

\[ A(-\infty) = 0, \quad A(+\infty) = \frac{2c_o}{\epsilon x}, \quad (7) \]

Straightforward but lengthy calculations show that the shock wave-like solution is possible only for supercritical base flow. The flow state establishing far downstream the transition is subcritical with the wave speed equal to \(-c_o\). The statement that the flow transition is possible only for the supercritical base flow, and that the final flow state is subcritical has been postulated by Benjamin [1]. Our result may be considered to be the proof of the Benjamin statement based on the analysis of the transition region. There is a deep physical analogy between described above transitions in swirling flows and shock waves in supersonic compressible flows. The formula (7) coupling the amplitudes of perturbations from both sides of the transition zone coincides with Rankine-Hugoniot conditions for weak shocks (Landau&Lifshitz [6]). The requirement to the flow to be supercritical upstream of the transition and subcritical downstream is identical to the requirement of the stability or the evolution condition for shocks (Landau&Lifshitz 6). Similar to the case of shocks the final flow state does not depend on the structure of the transitional region. This conclusion is particularly convenient in case if the extent of the transition region is much smaller than the
characteristic length of the flow. It allows avoiding the detailed calculations of this region with obscure physics. Our analysis gives conceptual insight into the physics of transition. It makes clear the desisive contribution of non-linearity of the inertia waves into transition. Similar to the theory of nonlinear waves (Whitham [16]) it is the wave non-linearity to be responsible for the deformation of the initially continuous wave profile, its subsequent break and, finally, to jump formation.

Qualitative behaviour of the amplitude $A$ as a function of the axial coordinate $z$ is shown in Fig.1. Far upstream and downstream the amplitude acquires its asymptotic values (7). In the transient zone the amplitude $A$ oscillates. The number of oscillations depends on the ratio of the viscous and the dispersive terms in the equation (6). The stream function of the columnar flow downstream is defined by the formulae (4), with $A$ given by eqn (7). The straightforward calculations show that the columnar flow establishing far downstream satisfies steady non-viscous Euler equations (2) with the same distribution of enthalpy $H(\Psi)$ and circulation $\Gamma(\Psi)$ as for the approach base supercritical flow. This is in fact the proof of the other basic postulate of Benjamin’s theory. Remind that according to Benjamin [1] the approach flow undergoes a transition leading to downstream axisymmetric “conjugate” subcritical flow sharing the same radial boundaries, mass flow, angular momentum distribution, and stagnation pressure distribution. In fact we did prove even more, i.e. that for a wide class of base supercritical flows the conjugate state does exist.

![Figure 1: Amplitude behaviour in transition region.](image)

There is another important issue that causes critics of Benjamin’s vortex breakdown theory [1]. According to Benjamin the flow-force increases due to transition that seems to be unphysical. In order to avoid this controversy Benjamin claimed in the paper [1] and the following papers [2], [3] that there is an unsteady mechanism due to wave resistance that provides conditions for flow force drain and allows avoiding this unphysical conclusion. The calculations based on the developed above theory show that flow force remains constant after
transition. It means that the mechanism of transition suggested in our paper is self-consistent and does not need involvement of additional physical processes into consideration. However, we should keep in mind that the suggested theory is, strictly speaking, valid only for weak transitions with the accuracy $\varepsilon^2$, and the flow-force problem might be shifted to the higher approximations. Remind in this connection the similar situation with weak shocks in compressible gas dynamics. The entropy and the stagnation pressure remain constant in the weak shocks with the accuracy $\varepsilon^2$. The change in these parameters appears to be of the order of $\varepsilon^3$ (see for example Landau & Lifshitz [6]), where $\varepsilon = M - 1$ ($M$ - Mach number). Accordingly, the extension of the developed in our paper theory to $\varepsilon^3$ and higher orders accuracy seems to be very desirable. It is worth to emphasize that the formula (4.13) from Benjamin [1]) is definitely not valid in our case. It predicts the nonzero flow-force growth within the accuracy $\varepsilon^2$ that as has been proven above, is not the case.

To illustrate the formulated above results the numerical calculations have been performed. The base flow has been modelled by Q-vortex (Leibovich [8,9]).

$$W(y) = W_1 + W_2 \exp(-\alpha y), \quad \Gamma_s(y) = V_1 [1 - \exp(-\alpha y)]$$

The parameters of the Q-vortex have been chosen close to operational conditions of the ALSTOM burners, where the flame is stabilized due to a vortex breakdown:

$$W_1 = W_2 = 15 \text{ m/s}, \quad \alpha = \frac{25}{R^2}, V_1 = x \cdot 30 / \sqrt{\alpha}, R = 0.3 \text{ m}$$

The parameter $x$ was being varied in the calculations. Equation (4) with the boundary conditions

$$\Psi(0) = 0, \quad \Psi(R^2) = \frac{1}{2} \int_0^{\pi} W(y) dy$$  \hspace{1cm} (8)

was solved using second order centred finite differences. The flow functions $H(\Psi), \Gamma(\Psi)$ entering the equation (2) were defined on the basis of Q-vortex. The calculations showed that in agreement with Benjamin [1] the boundary condition problem formulated above has in general several solutions. For each solution the wave speeds $c_s$ have been calculated by solving the eigenvalue problem (Leibovich [7]). The results of calculations are summarized as follows. There are two solutions of the boundary condition problem (2), (8) for $x < 1.45$. The first solution corresponding to supercritical flow is the Q-vortex base flow. The second one is the subcritical flow corresponding to Benjamin’s conjugate state. The Q-vortex base flow becomes subcritical for $x > 1.45$. Along with base
Q-vortex more than two supercritical solutions exist for this range of $x$. As it has been proved in the article the transition from supercritical state to subcritical is not possible at least for weak transitions. Therefore, an opposite to the range $x < 1.45$ situation takes place for $x > 1.45$. In this case one of supercritical solutions represents the approach flow in terms Benjamin’s formulation. The subcritical Q-vortex appears to be the conjugate state in the sense of Benjamin.

The wave speed $c_o$ both for the supercritical and the subcritical conjugate flows is plotted versus $x$ in the Fig 2. In accordance with the results of our analysis there are two values $c_o$ - positive and negative for given circulation. For $x < 1.45$ the positive $c_o$ corresponds to the Q-vortex, and for $x > 1.45$ the negative $c_o$ corresponds to the Q-vortex. Q-vortex curve in the figure is marked by 1. The calculations in the Fig.2, confirm theoretical result that the wave speed in the subcritical conjugate state equals to the speed of the approach flow with negative sign.

![Figure 2: Dependence of the wave speed from circulation.](image)

![Figure 3: Axial velocity profiles.](image)
The numbers 1, 2, 3 mark in the Fig. 3 the axial velocity profiles for the conjugate states corresponding to $x = 1.43$, $x = 1.3$, $x = 1.2$. With the reduction of swirl, the axial velocity on the axis drops. According to the calculations for $x < 1.05$ there is no solution with positive axial velocity on the axis. In this range of $x$ the flows with recirculation zones arise. The methods to extend the model for these flows have been discussed by Leibovich and Kribus [11], Wang and Rusak [15] and are out of the scope of the present study.

3 Conclusions

We study in this paper the nonlinear wave propagation in transcritical incompressible axisymmetric swirling flows. The necessity to have an observable, compact and reasonably simple picture of the processes responsible for vortex breakdown to control this phenomenon in gas turbine combustors with swirl stabilized flames has motivated the study. We show that under assumption of dissipation mechanism the transition between two columnar flow states is described by single Korteweg-de Vries-Burgers wave equation. Analyzing the equation we confirmed the validity of Benjamin theory. We showed that the final (or conjugate) columnar flow state is subcritical and governed by Long-Squire equation with the same distribution of total head and circulation as the base approach supercritical flow. We showed that the controversial conclusion of Benjamin’s theory concerning the increase the flow force (the axial momentum flux) due to transition is avoided in our consideration and. We showed in fact that the subcritical conjugate flow state exists for a wide class of supercritical approach flows.

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References