An inverse method for turbomachinery cascades of blades – investigation of the existence and uniqueness of solution

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Abstract

Inverse solution existence and uniqueness issues are discussed for a transonic two-dimensional inverse design method for turbomachinery cascades of blades. The prescribed conditions are the aerodynamic loading and the blade thickness. Afterwards we seek, in an iterative way, the blade geometry for the specified conditions. In order to accomplish this task we make use of a central finite-volume explicit time-marching scheme, to solve the Euler system of equations in 2D. This problem can sometimes be ill-posed. Theoretical and numerical evidence shows that we cannot always obtain a cascade of blades from an arbitrary distribution of load. Besides the problem of existence there is the problem of uniqueness. Both problems will be investigated in the context of the inverse method herein presented.

1 Introduction

The most recent developments occurring in the design of gas turbines and compressors do not merely seek an increase in the efficiency of turbomachines but also intend to minimize human involvement in order to reduce costs and error occurrence probability. One of the techniques that as gained increased attention for the design of turbomachinery cascades is based on inverse methods. These methods are based on the imposition of the pressure distribution needed to attain certain desirable conditions for the flow in the cascade of blades.
Numerical methods are then used in order to find the blade geometry that, for the desired conditions, will give the desired pressure distribution [1,2,3].

Inverse methods require that the user have some design experience and appreciate the subtlety of aerodynamic design in order to specify a pressure distribution for which a geometry exists. Aerodynamically attractive pressure distributions may require a physically unrealistic geometry or, in general, a geometry that is unattractive from the structural engineering standpoint. Because of the importance of the specification of the target pressure distribution in the inverse design problem, the issue of whether a unique solution (body geometry) exists for the given target pressure distribution must be considered in order to develop practical aerodynamic design methods.

Lighthill [4] first raised the inverse solution existence and uniqueness problem. Using conformal mapping techniques in two-dimensional incompressible flow, he demonstrated that a unique and correct solution to the inverse problem does not exist unless the prescribed pressure (or, equivalently speed) distribution satisfies certain integral constraints. Two constraints arise from the requirement that the airfoil profile have a specific trailing-edge gap. A third and subtler constraint requires that the prescribed surface speed distribution be compatible with the specified free-stream speed.

Volpe and Melnik [5] have demonstrated that the role of constraints and the question of correct formulation of the inverse problem have never been properly addressed for two-dimensional compressible flow and that, as a consequence, most existing inverse methods for transonic flow are not well formulated. Constraints similar to those for incompressible flow plausibly exist for the compressible flow condition. The existence of these three constraints implies that in general the target speed distribution must contain three free parameters to guarantee that the constraints can be satisfied through proper adjustment of the parameters. Thus, the originally prescribed target pressure changes, during the inverse design iterations, to satisfy the constraints. Introducing a potential function to formulate these three constraints mathematically, Volpe and Melnik successfully applied the method to two-dimensional compressible potential flow.

More recently Daripa [6] has also presented an analysis for the inverse problem in compressible flow that allows us to infer on the existence of solution under a compatibility condition. Is analysis allows to check for solution existence even for the case of over-specified problems. Daripa has converted the system of non-linear differential equations into a Beltrami equation valid for potential compressible flow. He concentrated on the third constraint of Lighthill. The free-stream Mach number obtained from the input pressure distribution using the irrotational and isentropic relations and Bernoulli’s equation is compared with \( M_C^\infty \) (The computed free-stream Mach number that is determined by the solution of the governing equations subject to the target pressure distribution). For a solution to exist, \( M_C^\infty \) determined directly from the input \( C_p \) and the computed \( M_C^\infty \) must be the same. Otherwise, a solution with the input pressure distribution does not exist. The other two constraints connected to the trailing edge closure problem were not discussed. Nevertheless the transonic
problem, with shock waves, where the Beltrami equation is no longer valid, remains analytically unsolved.

Even though some inverse methods for the two-dimensional Euler equations have been developed for transonic speeds, the existence and uniqueness problems of the inverse solutions have not been adequately studied because of the mathematical complexities in the formulation – Lee and Mason [7]. Due to the lack of proper mathematical formulation for compressible rotational flow Zannetti [8] studied this problem by using an empirical approach. He tackled this problem by systematically studying the behaviour of his inverse method under different constraints.

The purpose of this paper is to explore the existence and uniqueness of solution in the context of our transonic inverse method herein presented.

2 Presentation of the inverse method

There are different ways in which the inverse method can be formulated. The main differences are concerned with the choice of the prescribed variables. The classic design condition is the imposition of the pressure distribution in the blade surface; Léonard and Van den Braembussche [9] present this kind of approach. A different approach, that increasingly conducted to stable and robust results, is the one that gives the blade from a prescribed thickness distribution and camber line [1,10]. The camber line is obtained from the mean tangential velocity (that directly relates to the aerodynamic load) this together with the thickness distribution allow us to construct the cascade of blades. The design procedure is composed of two fundamental components:

1) An analysis code (coupled to a mesh generator).
2) An algorithm that changes the blade geometry (camber line).

The flow analysis algorithm makes the intensive component of the work. A change in geometry is achieved by a change in the camber line, \( y_{cl} = y_0 + f \), until convergence is attained, as is seen in figure 1.

The distribution of the mean tangential velocity component, \( \overline{V_y}(x) \), through the cascade is chosen as a design variable. This one is directly related to the force, or pressure load \( \Delta p(x) \), that the flow exerts on the blades. The relation between the mean tangential velocity and the camber line is given by:

\[
\frac{df^{N+1}}{dx} = \frac{df^N}{dx} + K \left( \frac{\overline{V_y^*}}{\overline{V_y}} - \frac{\overline{V_y^N}}{\overline{V_y}} \right) + \frac{1}{4} \frac{dt}{dx} \left( \frac{\Delta V_x^N}{\overline{V_x}} - \frac{\Delta V_x^{N-1}}{\overline{V_x}} \right);
\]

with,

\[
\overline{V_y}(x) = \frac{\int_{y_p}^{y_s} \rho u dy}{\int_{y_p}^{y_s} \rho dy}; \quad \overline{V_x}(x) = \frac{\int_{y_p}^{y_s} \rho u dy}{\int_{y_p}^{y_s} \rho dy}.
\]
The second term in eqn (1) is a correction that takes into account the thickness distribution. $\Delta V_x$ is the jump in axial velocity between the upper and lower surface.

The calculation of a new value for the camber line $(N+1)$ comes from the difference between the mean tangential velocity value of the previous iteration $\bar{V}_y^N(x)$ and the imposed value $\bar{V}_y^*(x)$, and also from a correction component due to the finite thickness of the blade $t(x)$. The method is presented in detail in Páscoa et al. [3]. Figure 1 shows that we must perform a flow analysis in each of the cascade of blades obtained by the inverse method until we reach convergence. This is typically attained when geometrical difference between the blade of iteration $N$ and $N+1$ is very small and around 1%.

As stated above the design method works by using, in each one of the design iterations, the information obtained from the analysis of the flow in the cascade of blades. In this method we solve the Euler system of equations in order to circumvent the limitation of certain methods to pure subsonic or supersonic flow, this is of paramount importance to account for the change of entropy through the shock waves. The analysis code (direct) is based on an improved time-marching Finite Volume code described in detail in Páscoa et al. [11]. The code has shown remarkably good shock capturing properties. It is an explicit time-marching method that solves the Euler equations in 2D with a finite volume central spatial discretization. The method includes a state-of-the-art artificial viscosity
component based on the limiter theory. This later has enhanced both the convergence and shock capturing properties of the code.

3 Existence and uniqueness of solution

3.1 Boundary conditions and the problem of the existence of solution

The existence of solution presumes that there is compatibility between the required load for the blade and the incoming flow – Daripa [6]. In the Euler flow analysis module of the code we impose boundary conditions based on the characteristics theory. For subsonic flow at entrance we should specify stagnation pressure $P_{01}$, stagnation temperature $T_{01}$ and the entrance flow angle $\alpha_1$ with the velocity extrapolated from the computational domain. For subsonic exit flow the static pressure $p_2$ is usually imposed and the three remaining primitive variables are extrapolated from inside the computational domain.

For a design task in which the required load is obtained from the analysis of a previous existing cascade there is no problem with the existence of solution. In that case there is compatibility between the imposed load for the blade and the incoming flow.

However, for design tasks in which an arbitrary load distribution is imposed we cannot always reach a converged solution, which is only attained in an approximate way. On the other hand in this method, and in opposition to methods by other authors [12], we impose the blade thickness as initial condition and therefore we don’t need to worry about reaching a realistic blade (with a closed trailing edge and a positive thickness), since this condition is always meted. For the case were there is no compatibility between the aerodynamic load and the incoming flow the method will give us the blade geometry that is the closest to the required one.

3.2 Analysis of uniqueness of the solution

For steady flow the tangential force acting on a blade can be related to the tangential component of the velocity upward and downward of the cascade of blades, $F = \dot{m}(v_1 - v_2)$. As referred by Zannetti [8] $F$ changes as a function of exit angle $\alpha_2$, see figure 2. Starting from a set of values for $P_{01}$, $T_{01}$, $\alpha_1$ and $p_2$ we can determine $F(\alpha_2)$ for a general cascade. The data of figure 2 was obtained analytically through the homoentropic relations. It is clearly seen that $F$ will be zero for three values of $\alpha_2$. In $\alpha_2 = \pm 90^\circ$ the value of $u_2$ and as a consequence $\dot{m}$ will vanish. In these two points the blades are completely deflected at the trailing edge and therefore the flow doesn’t have any axial component of the velocity. Besides, $F$ will also vanish for $\alpha_2 = \alpha_1$ that is when the blade doesn’t deflect any flow. The force is positive for $\alpha_2 < \alpha_1$ and vice-versa.

Now we consider the inverse problem that must be solved by imposing a set of boundary conditions ($P_{01}$, $T_{01}$, $\alpha_1$ and $p_2$) and a load distribution.
\( \Delta p(x) \), or mean tangential velocity \( \overline{V_t}(x) \). On the base of the previous considerations we can conclude that there will be two different geometries of the blade that can meet the required conditions or else any.

The force \( F \) can be determined by \( F = \int \Delta p(x) \, dx \), with the integral taken along the \( x \)-axis between the leading and trailing edge of the blade. For \( F > F_{\text{max}} \) or \( F < F_{\text{min}} \) there is no solution, but for \( F_{\text{min}} < F < F_{\text{max}} \) there will be two cascades of blades that can provide the same force (as is seen for A and B in figure 2). Therefore we can conclude that the same force can be obtained through a lower mass flow and greater deflection, as in case A, or by a higher mass flow and lower deflection as in case B.

![Figure 2: Evolution of the force \( F \) as a function of \( \alpha_2 \).](image)

### 4 Practical behaviour of the inverse method

In order to check for the behaviour referred in figure 2 we have tested our inverse method in the re-design of a cascade consisting of NACA0012 blades. The considered pitch/chord ration is 0.5. The conditions specified for the inflow are \( Ma_1 = 0.55 \) with \( \alpha_1 = 5^\circ \). In figure 3 we can see the evolution of the force obtained through multiple analysis of the flow in the cascade for different blades with exit angles \( \alpha_2 \). The evolution of the graph is quite similar to that presented in figure 2. For this cascade we can see from figure 3 that with \( \alpha_2 = -20^\circ \) we can obtain the same force that is obtained with \( \alpha_2 = -35^\circ \). We may also conclude that the mass flow for \( \alpha_2 = -35^\circ \) is 79\% lower than the one obtained with \( \alpha_2 = -20^\circ \).

Using the results from the analysis presented in figure 3 we have run the code in design mode in order to redesign the two cascades of blades defined for an exit angle of \( \alpha_2 = -35^\circ \) and \( \alpha_2 = -20^\circ \).
Figure 3: Evolution of the force as a function of $\alpha_2$ computed with the Euler analysis module of the inverse method.

Figure 4: Evolution of the position of the trailing edge through the design iterations.

In figure 4 we can see the evolution of the position of the trailing edge through the design process. It is clear that the cascade of blades for which $\alpha_2=-20^\circ$ shows a very monotone evolution for the position of the trailing edge that actually reaches convergence after nine design iterations. For the cascade with $\alpha_2=-35^\circ$ the behaviour of the trailing edge position is quite oscillatory, as seen from figure 4. Figure 5 shows the evolution of the cascade of blades during the inverse design.
Figure 5: Evolution of the flow-field during the design iterations. a) First design iteration for $\alpha_2=-20^\circ$. b) Second design iteration for $\alpha_2=-20^\circ$. c) Last design iteration for $\alpha_2=-20^\circ$. d) First design iteration for $\alpha_2=-35^\circ$. e) Second design iteration for $\alpha_2=-35^\circ$. f) Last design iteration for $\alpha_2=-35^\circ$. 
5 Conclusions

In this work we have presented an analysis on the existence and uniqueness of solution for an inverse design method for turbomachinery cascades. There are no analytical solutions for the problem of existence and uniqueness of solution in rotational transonic compressible flow, therefore we have used an empirical but quite formal approach to analyse this problem in the particular case of our inverse method.

During this study it became clear that our inverse method can always come up with a realistic blade and can provide a solution, at least approximately, in the case were there is no compatibility between the incoming flow and the prescribed load.

From the results obtained we may conclude that for a lower stagger angle the method converges monotonically to the target cascade of blades. For higher stagger angles, particularly for stagger angles greater than the point of inflexion which corresponds to the maximum force obtained from the cascade, the method shows an oscillatory behaviour during convergence to the target cascade.

After performing the same analysis on different cascades, under different flow conditions, we have seen the non-monotone convergence behaviour. This happens always that we redesign a cascade for a stagger angle greater than the point of inflexion. The same behaviour has been detected by other authors [12] using a different formulation for the inverse method.

A deeper understanding of this behaviour could lead to a faster convergence of the inverse method; this is of paramount importance for its extension to 3D.

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References


