Progressive waves in channels with some specific cross sections

C. -H. Kong & C. -M. Liu
Department of Naval Architecture and Ocean Engineering,
National Taiwan University, Taiwan.

Abstract

Progressive surface waves in channels with some specific cross sections, which include trapezoid and elliptic shapes, are studied analytically in our present paper. Some analytical solutions of the frequencies of longitudinal modes of progressive surface waves are provided. The convergence of the eigenvalues obtained by different methods is well compared and discussed.

1 Introduction

Solutions for progressive surface waves in open channels of variable depth are important to hydrographers and engineers concerned with wave motion in rivers. The number of explicit analytical solutions for wave motion in open channels of variable depth is extremely small. The solutions for the frequencies of water in open channels with certain triangular sections presented in Lamb’s work [1] remain to this day the only exact linear solutions in existence for surface waves. In all the cases of surface waves under investigation, the governing mathematical system represents an eigenvalue problem. The eigenvalue is the wave frequency, and the eigenfunction is either the velocity potential or the stream function, or else some other function which is an unknown function of the mathematical system. It is the purpose of this paper to determine the eigenvalues and the eigenfunctions for the cases under study here. The methods of solution and the results will be presented in the following sections.

In this paper, gravity waves propagate along the longitudinal axis with various cross sections will be studied. The wave length is also along the longitudinal axis of the channel. Thus we shall call the wave motion a longitudinal mode. The progressive surface waves in either a elliptic channel or a trapezoid channel may...
also be under the influence of transverse oscillation, in a to-and-fro fashion. If so, the longitudinal waves should include this motion.

2 The case of trapezoid channel

2.1 Formulation of the problem
If viscous effects are ignored and the wave motion is supposed to have started from rest, since density of the water is assumed constant, the motion is irrotational and a velocity potential \( \Phi \) exists. The gradient of \( \Phi \) is the velocity vector. We use the Cartesian coordinates \((x, y, z)\) with \(z\) measured longitudinally, \(y\) measured vertically, and \(x\) measured across the channel. The Laplace equation is

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0
\]  

(1)

At solid boundaries the normal velocity component vanishes, so that

\[
\frac{\partial \Phi}{\partial n} = 0
\]  

(2)

where \(n\) is measured in a direction normal to the solid boundaries. The condition eqn (2) at the free surface is

\[
\frac{\sigma^2}{g} \Phi = \frac{\partial \Phi}{\partial y}
\]  

(3)

Consider the channel with width \(2b\) and let

\[
(\hat{x}, \hat{y}, \hat{z}) = \left( \frac{x}{b}, \frac{y}{b}, \frac{z}{b} \right)
\]

After dropping the circumflexes, eqn (1) and eqn (2) remain unchanged, but eqn (3) becomes

\[
\frac{\sigma^2 b}{g} \Phi = \frac{\partial \Phi}{\partial y}
\]  

(4)

We assume the velocity potential to have the form

\[
\Phi = \Phi'(x, y)e^{i(kz - \alpha)}
\]  

(5)

Figure 1: Definition sketch for trapezoidal cross-section channel
where \( k \) is the dimensionless longitudinal wave number equals to \( 2\pi \) over dimensionless wave length. Henceforth \( \Phi \) stands for \( \Phi(x,y) \) on the right-hand side of eqn (5), unless otherwise stated. Substituting eqn (5) into eqn (1), we have

\[
\frac{\partial \Phi}{\partial x^2} + \frac{\partial \Phi}{\partial y^2} - k^2 \Phi = 0
\]  
(6)

From eqn (2) and eqn (5), we have

\[
\frac{\partial \Phi}{\partial n} = 0 \quad \text{or} \quad \left( \frac{\partial \Phi}{\partial x} i + \frac{\partial \Phi}{\partial y} j \right) \cdot \hat{n} = 0
\]  
(7)

where \( \hat{n} \) is a unit vector normal to the solid boundaries. At the free surface

\[
\lambda \Phi = \frac{\partial \Phi}{\partial y}
\]  
(8)

in which

\[
\lambda = \frac{\sigma^2 b}{g}
\]  
(9)

The differential system to be solved consists of eqns (6), (7) and (8), with \( \lambda \) as the dimensionless parameter.

Consider a channel of trapezoid cross section, and set the unit width and the depth of the channel to be unity and \( h \) respectively, as shown in Fig. 1. The following two analytical solutions for longitudinal modes in a trapezoid channel are introduced.

### 2.2 The Bessel-function and hyperbolic-function solutions

Two method to solve the eigenvalues of gravity waves propagating in trapezoid channels are provided herein. First, the Bessel-function method is applied to solve the problem. Consider the cylindrical coordinate system, the Laplace equation can be written as

\[
\frac{\partial \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0
\]  
(10)

We assume the solution of eqn (10) is of the separation form as

\[
\Phi = W(r) \cdot Q(\theta) \cdot e^{i(kr - \sigma t)}
\]  
(11)

Then the Laplace equation can be separated as

\[
r^2 \frac{\partial^2 W}{\partial r^2} + r \frac{\partial W}{\partial r} - \left( k^2 r^2 + n^2 \right) W = 0
\]  
(12)

\[
\frac{\partial^2 Q}{\partial \theta^2} + n^2 Q = 0
\]  
(13)

According to the solutions of above equations, the velocity potential of even mode can be represented as

\[
\Phi = \sum_{n=0}^{N} \left[ A_n I_n(kr) + B_n K_n(kr) \right] \cos(n\theta) \exp[i(kz - \sigma t)]
\]  
(14)

with
In eqn (14) \( I_n(r) \) and \( K_n(r) \) indicate the modified Bessel functions. Similarly, the potential of odd mode is

\[
\Phi = \sum_{n=1}^{N-c} \left[ A_n I_n(kr) + B_n K_n(kr) \right] \sin(n\alpha) \exp\left[i(kz - \omega t)\right]
\]

(16)

with

\[
n = \frac{(2m-1)}{2\alpha} \quad m = 1, 2, 3, ..., M
\]

(17)

It is obvious that eqns (14) and (16) not only satisfy the Laplace equation but automatically fit the side boundary condition

\[
\frac{\partial \Phi}{\partial \theta} = 0 \quad \text{at} \quad \theta = \pm \alpha
\]

(18)

Consider the free surface and bottom boundary conditions

\[
\lambda \Phi = \frac{\partial \Phi}{\partial r} \cos \theta - \frac{\partial \Phi}{\partial \theta} \frac{\sin \theta}{r} \quad \text{at free surface}
\]

(19)

\[
\frac{\partial \Phi}{\partial y} = 0 \quad \text{at bottom}
\]

(20)

in which the eigenvalue \( \lambda \) is of the form shown in eqn (8). One can insert eqns (14) and (16) into eqns (19) and (20) respectively to solve the eigenvalues. Similar to the Bessel-function method, the solutions of odd and even mode combined with hyperbolic functions are shown respectively

\[
\Phi = \sum_{n=1}^{N} \left[ A_n \cosh(x \sin \alpha) \cos(y \cos \alpha) + B_n \cosh(y \cos \alpha) \cos(x \sin \alpha) \right] \cos(n\alpha) e^{i(kz - \omega t)}
\]

(21)

with

\[
n = \frac{m\pi}{\alpha} \quad m = 0, 1, 2, ..., M
\]

(22)

and

\[
\Phi = \sum_{n=1}^{N} \left[ A_n \cosh(x \sin \alpha) \cos(y \cos \alpha) + B_n \cosh(y \cos \alpha) \cos(x \sin \alpha) \right] \sin(n\alpha) e^{i(kz - \omega t)}
\]

(23)

with

\[
n = \frac{(2m-1)}{2\alpha} \quad m = 1, 2, 3, ..., M
\]

(24)

We also can solve the eigenvalues of eqns (21) and (23) by inserting them into two boundary conditions eqns (19) and (20).

### 2.3 Discussion

To solve the eigenvalues, the beginning \( N \) terms of eqns (14), (16), (21) and (23) can be arbitrarily selected. Hence the corresponding \( N \) eigenvalues can be
solved by inserting these $N$ terms into boundary conditions eqns (19) and (20). Table 1 shows the results of Bessel-function solutions. The cases of sloping angle $45^\circ$ is discussed herein. It is logically reasonable that the more terms we take, the more accurate eigenvalues we’ll obtain. In other words, the eigenvalues will converge to a specific value if $N$ is large enough. Table 2 represents the results of hyperbolic-function solutions. It is also needed to emphasized that the solution consisted of hyperbolic functions is only applicable to the case of the sloping angle $45^\circ$ of trapezoid channels. Fig.2 represents the relation between eigenvalues and $k$ for $H=0.25$ and $H=0.5$. Each eigenvalue will approach to a specific value as $k$ approaches zero. Comparing the convergence of these two solutions on the angle $45^\circ$, Fig.3 shows the convergence of different eigenvalues for the case $h=0.5$ and $k=0.5$. It is clear that the adoption of hyperbolic functions causes the faster convergence happening almost for each eigenvalue. Besides, it also shows in each case that the eigenvalue will converge to a specific value while $N \geq 5$.

Table 1: The eigenvalues for the first four modes of waves propagated along the longitudinal axis of trapezoid channels (Bessel-function solutions)

<table>
<thead>
<tr>
<th>$h$</th>
<th>$k$</th>
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<th>$N=3$</th>
<th>$N=4$</th>
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<td>0.5</td>
<td>$\lambda_0$</td>
<td>0.0920</td>
<td>0.0923</td>
<td>0.0921</td>
<td>0.0920</td>
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<td></td>
<td>$\lambda_1$</td>
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<td>0.961</td>
<td>0.954</td>
<td>0.953</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2$</td>
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<td>2.347</td>
<td>2.343</td>
<td>2.343</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_3$</td>
<td>3.684</td>
<td>3.888</td>
<td>3.879</td>
<td></td>
</tr>
<tr>
<td>0.25</td>
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<td>$\lambda_0$</td>
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<td>0.0549</td>
<td>0.0547</td>
<td>0.0545</td>
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<tr>
<td></td>
<td></td>
<td>$\lambda_1$</td>
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<td>0.641</td>
<td>0.629</td>
<td>0.623</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>0.00375</td>
<td>0.00375</td>
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<td></td>
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<td></td>
<td></td>
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</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_3$</td>
<td>3.313</td>
<td>3.587</td>
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<td>3.513</td>
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</table>
Table 2: The eigenvalues for the first four modes of waves propagated along the longitudinal axis of trapezoid channels (hyperbolic-function solutions)

<table>
<thead>
<tr>
<th>h</th>
<th>k</th>
<th>N=1</th>
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<th>N=4</th>
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<td></td>
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<td>0.953</td>
<td>0.953</td>
<td>0.953</td>
<td></td>
</tr>
<tr>
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<td></td>
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<td>2.343</td>
<td>2.343</td>
<td></td>
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<tr>
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<td></td>
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<td>3.879</td>
<td>3.879</td>
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</tr>
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<td>1.942</td>
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<td>3.513</td>
<td>3.513</td>
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</table>

Figure 2: The eigenvalues for the case (a) $h=0.5$ (b) $h=0.25$
Figure 3: The convergence of the eigenvalues of the first four modes. Bessel-function method (dash line), hyperbolic-function method (solid line).

Figure 4: Definition sketch for elliptic cross-section channel

3 The case of elliptic channel

3.1 Formulation of the problem and the solution
Consider a channel of elliptical cross-section shown in Fig.4, we define the sketch of the cross-section as

\[ y = -h \sqrt{1 - x^2} \]  \hspace{1cm} (25)

And the free surface is at

\[ y = -d' = -(h - d) \]  \hspace{1cm} (26)

in which all quantities are dimensionless. The velocity potential is assumed to have the form
Then the Laplace equation becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - k^2 \phi = 0$$

For the odd modes, we assume

$$\phi(x, y) = \sum_{n=1}^{\infty} \sin k_n x \left( A_n e^{p_n y} - B_n e^{-p_n y} \right)$$

in which

$$k_n = \frac{2n - 1}{2} \pi \quad n = 1, 2, \ldots, N$$

Substituting eqn (29) into eqn (28), we have

$$p_n = \sqrt{k_n^2 + k^2}$$

At solid boundaries the normal velocity component vanish, the condition is completely the same as eqn (7)

$$\left( \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right) \cdot \hat{n} = 0$$

where

$$\hat{n} = \frac{\sqrt{x^2 + (y/h^2)^2}}{V}$$

which is measured in a direction normal to the solid boundary. eqn (31) can be reduced to

$$x \frac{\partial \phi}{\partial x} + \frac{y}{h^2} \frac{\partial \phi}{\partial y} = 0$$

To satisfy the condition of eqn (32) on the solid boundary, we substitute eqn (29) into eqn (32) and obtain

$$\sum_{n=1}^{N} \left[ \left( x k_n \cos k_n x e^{p_n y} + \frac{y}{h^2} p_n \sin k_n x e^{-p_n y} \right) A_n \right. \\
- \left. \left( -x k_n \cos k_n x e^{p_n y} + \frac{y}{h^2} p_n \sin k_n x e^{-p_n y} \right) B_n \right] = 0$$

where $$y = -h \sqrt{1 - x^2}$$ . Substituting eqn (29) into the free surface condition, $$\lambda \phi = \phi_{y'}$$ , and putting $$y = -d'$$ , we have

$$\sum_{n=1}^{N} \left[ (\lambda - p_n) e^{-p_n d'} A_n - (\lambda + p_n) e^{p_n d'} B_n \right] \sin k_n x = 0$$

or

$$\begin{align*}
(\lambda - p_1) e^{-p_1 d'} A_1 - (\lambda + p_1) e^{p_1 d'} B_1 &= 0 \\
(\lambda - p_2) e^{-p_2 d'} A_2 - (\lambda + p_2) e^{p_2 d'} B_2 &= 0 \\
&\vdots \\
(\lambda - p_N) e^{-p_N d'} A_N - (\lambda + p_N) e^{p_N d'} B_N &= 0
\end{align*}$$
Multiplying eqn (33) by \( \sin k_n x \), and integrating between \(-1\) and \(1\) in which
\[
k_m = \frac{2m - 1}{2} \pi, \quad m = 1, 2, \ldots, N
\]
we obtain, together with eqn (35), \(2N\) homogeneous equations in \( A \)'s and \( B \)'s. The vanishing of the determinant of the coefficients provides the secular equation which determines \( \lambda \) then the \( A \)'s and \( B \)'s. Similarly, we may assume \( \phi(x, y) \) for the even longitudinal mode to be
\[
\phi(x, y) = \sum_{n=1}^{\infty} \cos k_n x \left( A_n e^{p_n y} + B_n e^{-p_n y} \right)
\]
where
\[
k_n = 2\pi, \quad n = 0, 1, 2, \ldots, N
\]
Using the same process as that described for the odd mode, we can determine \( \lambda \) for the even mode. If eqns (29) and (36) are applied to measure the free surface elevation, \( y = \eta(x, y, t) \), we have
\[
\eta = -\sum_{n=1}^{N} \frac{p_n}{\sigma} \sin k_n x \left( e^{-p_n k z} A_n + e^{p_n k z} B_n \right) \sin k z \quad \text{for the odd mode} \quad (37)
\]
and
\[
\eta = -\sum_{n=1}^{N} \frac{p_n}{\sigma} \cos k_n x \left( e^{-p_n k z} A_n - e^{p_n k z} B_n \right) \sin k z \quad \text{for the even mode} \quad (38)
\]

### 3.2 Discussion

The frequencies for the first two longitudinal modes in half-filled elliptical channels with various values of \( h \) are given in Tables 3 and 4, as well as in Fig. 5. For the gravest even mode and for the case of \( k = 0.5 \) and \( k = 1.0 \), \( \lambda \) has been computed for various values of the number of terms \( N \) ranging from 3 to 20. It has been shown in each case that \( \lambda \) converges to a definite value, reached (practically) as \( N > 15 \). All results are displayed in Table 3 and 4.

**Table 3:** The eigenvalues of the gravest even mode of waves propagating along the longitudinal axis of elliptical channels

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<tr>
<th>Items</th>
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<th>( k = 1.0 )</th>
</tr>
</thead>
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<td>( h = 1 )</td>
<td>( h = 0.5 )</td>
</tr>
<tr>
<td>( N )</td>
<td>( \lambda )</td>
<td>( \lambda )</td>
</tr>
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<td>3</td>
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Table 4: The eigenvalues of the first odd mode of waves propagating along the longitudinal axis of elliptical channels

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</table>

4 Conclusion

The solution of the frequency for any given mode is under the influence of transverse oscillation. The results for different modes of waves propagating in trapezoid channels with various the depth $h$ for any given $k$ are given by the methods of the Bessel-function solution and the hyperbolic-function solution described in this article. We note that the frequency of any mode increases with the depth and with the water occupied area, as it should. Comparing our two solutions, it is explicit that the hyperbolic-function solution has a better convergent efficiency than that of the Bessel-function solution. Also it has been shown for each case that all eigenvalues will converge to a specific value as $N \geq 5$. As for the elliptical case, most eigenvalues converges to a definite value as $N > 15$.

It is hoped that the specific results provided by this work will be useful to hydraulic engineers and hydrologists. Furthermore, it is evident that the method described in the foregoing will be applicable to solving problems of surface waves in any given channel with regular or irregular cross-sections.

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References