Toward a construction of a class of scheme based on wavelet decomposition

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Abstract

The paper proposes a new class of scheme based on wavelet decomposition. The field variables and the Navier-Stokes equations are projected onto the space spanned by the scaling functions. The scheme is first applied to the one-dimensional inviscid Burgers’ equation and the results show excellent agreements with the exact one. The one dimensional Euler equation with the Sod condition is also solved and the scheme captures the sharp shock front and contact discontinuity. The artificial dissipation terms are derived from physical analysis. The scheme, in addition, successfully solved the cavity flow problems with the Reynolds number up to 20000 with high accuracy. In the final part, a connection between the artificial dissipation and turbulence viscosity is discussed.

1 Introduction

Data analyses using wavelets and scaling functions are recently often used in image processing, signal processing and so on[1]. Not only some wavelets and scaling functions form a complete orthogonal sets, but also they are compactly supported as the shape functions used in FEM[2]. Therefore a scheme for non-linear equations utilizing the characteristics of wavelets has to be made and tried. The main advantage of the scheme is that the multi resolution analysis is applicable and simple and useful space such as the Hilbert space can be utilized. By using these characteristics of the wavelet, shocks capturing schemes and turbulent mixing models can well be constructed. In the paper, wavelet schemes for the one-dimensional inviscid Burgers’ equation and Euler equation are solved and the results are compared with the exact ones. Two-dimensional cav-
ity flow problems are also solved and the result are compared with Shrieber's result[3]. In the final part, the authors try to construct a quasi-turbulence model, and the results are examined.

## 2 Nomenclature

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<td>$u, v$</td>
<td>$x, y$ components of velocities</td>
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<td>$x, y$ components of velocity fluctuations</td>
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<tr>
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<tr>
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<tr>
<td>$\rho$</td>
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<td>$\psi$</td>
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### Suffix

<table>
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<th>Symbol</th>
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<tr>
<td>$i,j,k$</td>
<td>Indices of basis functions</td>
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<td>$N$</td>
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<td>$L,M$</td>
<td>Resolution Level $(\text{super script})$</td>
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## 3 Construction of scheme in one-dimensional space

### 3.1 Assumption for the basis

Let us introduce the following basis and use Dirac'bra-ket representation:

$$\text{Span}\left\{\phi_k, \psi_j^M ; k,j \in \mathbb{Z}, M \in \mathbb{Z}^+\right\} = L^2(R)$$

$$\text{supp}\left\{x \phi_k\right\} = [k, k+S]$$

$$\text{supp}\left\{x \psi_k^M\right\} = [k, k+2^{-M}S]$$

$$\langle \phi_k | \phi_j \rangle = \delta_{k,j}, \langle \phi_k | \psi_j^M \rangle = 0, \langle \psi_k^L | \psi_j^M \rangle = \delta_{k,j} \delta_{L,M}$$

$$\{x | \phi_k\}, \{x | \psi_k^M\} \in C^1$$

By imposing these conditions $\{\phi_k\}$ becomes a set of scaling functions and $\{\psi_k\}$ is the set of wavelets. Since the authors focus only on compactly supported scaling functions and wavelets, the authors assume the support $\phi_k$ and $\psi_k$ is $[k, k+S]$. 

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3.2.1 Expressing by scaling functions

Let us consider the following one-dimensional inviscid Burgers' equation:

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \] (6)

By applying the first order Euler discretization in time, we obtain:

\[ u^{n+1} = u^n - \Delta t u^n \frac{\partial u^n}{\partial x} \] (7)

If \( u^n \) is composed of \( N \) scaling functions:

\[ u^n(x) = \sum_{k=0}^{N-1} u_k^n \phi_k(x) \] (8)

Because of the nonlinear term in eqn(7), it takes infinite numbers of wavelets to express \( u^{n+1} \):

\[ u^{n+1}(x) = \sum_{k=0}^{N-1} u_{k+1}^n \phi_k(x) + \sum_{k=0}^{\infty} \sum_{i=0}^{M} u_{i+j}^n \psi_i^M(x) \] (9)

If it is approximated with the original scaling functions:

\[ \sum_{k=0}^{N-1} u_k^n \Delta t \frac{\partial u_k^n}{\partial x} = 0 \] (10)

Where \( D \) is defined as follows:

\[ \langle x | D^n g \rangle = f(x) \frac{\partial^n g(x)}{\partial x^n} \quad f, g \in L^2(R) \] (11)

The discretized Burgers' equation becomes:

\[ u_{k+1}^n = u_k^n - \Delta t \sum_{m,n} J_{m,n} u_{k+m,n} \] (13)

The size of \( J_{m,n} \) is about \( S \times S \) and it can be prepared before the simulation.

3.2.2 Instability detection and artificial dissipation

While the solution is very smooth, eqn(13) works very well. However any shock like structure develops, eqn(13) collapses. Therefore a TVD scheme like
procedures have to be done:
A. Detection of instabilities of the solution.
B. Increment of the artificial disposition according to the instabilities.

In order to perform procedure A, let us consider the moment theorem[1]:

\[ \int \psi^N(x)x^m \, dx = 0 \quad (m < N) \]  

Let \( f(x) \) be the nonlinear term of eqn(7), and apply Taylor expansion about \( (x-k) \).

\[ f(x) = \sum_{n=0}^{N-1} \frac{1}{n!} \frac{d^n f}{dx^n} (x-k)^n + R(x-k) \]  

\[ \int \psi^N(x-k) f(x) \, dx = \int \psi^N(x-k) R(x-k) \, dx \]  

If the nonlinear term becomes very steep like the step function near \( x = k \), \( R(x-k) \) in eqn(15) gets larger, thereby making the integration of eqn(16) large. Contrary if \( f(x) \) is nearly flat or linear, the integration becomes very small. This measure of the instability caused by the nonlinear term is described as follows:

\[ \Delta u_k = -\Delta t \int \psi(x-k) f(x) \, dx \]  

3.2.2 Derivation of Artificial Dissipation Term

To carry out the procedure B, artificial dissipation terms are derived from the measure of the fluctuation produced in one calculation step. From eqn(17):

\[ \tilde{u}^{n+1}_k = u_k - \Delta t \int \psi(x-k) u^n(x) \frac{\partial u^n(x)}{\partial x} \, dx \]  

By an analogy with the statistical physics of gases[5], we make the following assumptions: The fluctuation activates virtual randomly moving particles, which transport the man field values as they randomly move. The energy of the virtual particles are assumed as follows:

\[ e^{n+1}(x) = e^n(x) + \sum_k \tilde{u}^{n+1}_k \phi_k(x) \]  

From the assumption, the following viscous term is generated:

\[ \frac{\partial}{\partial x} \left( \lambda \sqrt{e^n} \frac{\partial u^n}{\partial x} \right) \]  

Where \( \lambda \) is the mean free path of the virtual particle. It must be determined by numerical experiments, however it is probably close to \( S \). In addition to the definition of the random particle energy, the following formulation is also possible:

\[ e^{n+1}_k = e^n_k + \Delta t \left( \phi_k u^n D e^n \right) = \Delta t \lambda \left( \phi_k \left( \left| \tilde{u}^n \right| D \right) e^n \right) + \sum_k \left| \Delta u_k \right|^2 \]  

where

\[ \left| \tilde{u}^n \right| (x) = \sqrt{\langle \left| \tilde{u}^n (x) \right|^2 \rangle} = \sum_k \sqrt{e^n_k} \phi_k(x) \]  

From here we use the following convention:
Wavelet coefficient of velocity field fluctuation generated in one time step.

Accumulated velocity fluctuation as a function of position.

Unlike the eqn(19), in eqn(21) the convection and diffusion of the random particle energy is taken into account. Which ever the random particle energy formulation is, the inviscid Burgers’ equation with the artificial dissipation becomes as follows:

\[ u^{n+1} - u^n + \Delta t u^n \frac{\partial u^n}{\partial x} = -\Delta t \frac{\partial}{\partial x} \left( \lambda \left| u^n \right| \frac{\partial u^n}{\partial x} \right) \]  

By projecting eqn(23) onto the space spanned by the scaling functions:

\[ u_k^{n+1} - u_k^n + \Delta t \left( \phi_k \left| u_k^n D u^n \right| - \Delta t \lambda \left( \phi_k \left| u_k^n \right| D u^n \right) \right) \]

The inner products in eqn(24) is calculated as follows:

\[ \langle \psi_k | f D g \rangle = \sum_{m,n} f_{k+m} g_{k+n} K_{m,n} \]

\[ \langle \phi_k | D (f D g) \rangle = \int \frac{\partial \phi(x-k)}{\partial x} \left( J(x) \frac{\partial g(x)}{\partial x} \right) dx \]

\[ = \sum_{m,n} f_{k+m} g_{k+n} D_{m,n} \]

3.2.3 Solving Burgers’ equation to validate the scheme

The scheme is applied to the one dimensional inviscid Burgers’ equation. The domain is [0,2] and the initial condition is \( u(x, t=0) = \sin(\pi x) \). As the basis, symlet-4 is chosen. The calculation time step is \( \Delta x/(20xU_s) \) and \( U_s \) is the maximum value of \( u \) at the initial state. Figure 1 shows the energy distribution and a comparison of the exact solution and numerical one, which is composed of 200 scaling functions. The numerical solution is a continuous function, however the dots are plotted to make compression easier. Despite asymmetrical shape of the basis, the solution is quite symmetric. The kink contains just one point. This means the thickness of the kink 1/(5x200) = 1/1400 and it is remarkable.

4 One dimensional Euler equations with sod condition

Next, the scheme is applied to the Euler equation with the Sod condition, which is often used to benchmark schemes. The initial condition is therefore:

In the left half of the space: \((\rho_L, u_L, p_L) = (1.0, 0.0, 1.0)\)

In the right half of the space: \((\rho_R, u_R, p_R) = (0.125, 0.0, 0.1)\)

The Euler equations with the artificial dissipation terms become as follows:
The ratio of specific heat $\gamma$ is set at 1.4. The artificial dissipation model described by eqn(19) is chosen and the following definition is used in eqns(27):

$$u(x) = \sum \sqrt{v(x)}$$

By projecting the second equation of eqns(27) onto the wavelet space, the equation of motion for the virtual random particle field is obtained:

$$\frac{\partial u_k}{\partial t} + \sum_{mn} \left( u_{k+m} u_{k+n} + \beta_{k+m} p_{k+n} \right) K_{mn} \lambda \sum_{mn} \tilde{u}_{k+m} u_{k+n} E_{mn}$$

with

$$E_{m,n} = \int \psi(x) \frac{\partial}{\partial x} \phi(x+m) \frac{\partial}{\partial x} \phi(x+n) dx$$

By projecting the whole eqns(27) onto the space spanned by the scaling functions, a spatially discretized set of equations are obtained:

$$\frac{\partial p_k}{\partial t} + \sum_{mn} \left( u_{k+m} p_{k+n} + \rho_{k+m} u_{k+n} \right) J_{mn} \lambda \sum_{mn} \tilde{u}_{k+m} p_{k+n} D_{mn}$$

$$\frac{\partial u_k}{\partial t} + \sum_{mn} \left( u_{k+m} u_{k+n} + \beta_{k+m} p_{k+n} \right) J_{mn} \lambda \sum_{mn} \tilde{u}_{k+m} u_{k+n} D_{mn}$$

$$\frac{\partial u_k}{\partial t} + \sum_{mn} \left( u_{k+m} p_{k+n} + \gamma p_{k+m} u_{k+n} \right) J_{mn} \lambda \sum_{mn} \tilde{u}_{k+m} p_{k+n} D_{mn}$$

where $\beta$ is the inverse of $\rho$. $\beta$ is calculated, so that it satisfies the following relation in the present sub space spanned by the scaling functions.

$$\beta(x) \rho(x) = 1$$

By multiplying a scaling function and integrating over the relevant region:

$$\sum_{i,j} \int \beta_i \rho_j \phi(x-k) \phi(x-i) \phi(x-j) dx = 1$$

We obtain the following iterative equation:

$$\beta_i = \frac{1}{\rho_{i,0,0} \left[ 1 - \sum_{m\neq 0, n\neq 0} \beta_{i+m} \rho_{i+n} I_{m,n} \right]}$$

After a few iteration, its error becomes less than $10^{-10}$.

Figure 2 shows numerical and exact solutions. As the basis, Daubechies-2 is chosen, $\lambda$ is set at 2 and 600 scaling functions are used. The time step is $\Delta x/(25xU_s)$ and $U_s$ is the theoretical shock propagation speed. The numerical
solutions perfectly agree with the exact ones in the expansion region, and the scheme also captures the shock and contact discontinuity very sharply. Figure 3 shows the density field obtained with symlet-4, and the results are essentially the same. The same simulation is carried out with Daubechies-3 to 5, all results are almost same. The contact discontinuity becomes, however, slightly thicker and smoother, when higher order Daubechies’s scaling functions are used.

Figure 4 shows the random particle energy distribution. The energy, or the artificial dissipation coefficient, has two extremely sharp peaks. One is located near \( x=0 \) where the expansion and the shock begin. There is another peak at the front of the shock. These peaks make the artificial dissipation larger and suppresses the instabilities at the two location. In other regions, the energy is almost zero, hence the dissipation terms work only where they are necessary.

As shown in figures 2 and 3, the numerically calculated shock front speed is slightly slower than exact one. The direct cause is the velocity behind the shock is smaller than theoretical one as seen in figure 2. The authors believe use of the characteristic method will solve this problem with costing CPU time. Another
question is that the random particle energy becomes negative near $x=0$. Eqn(19) does not guarantee positive definiteness of the energy, but no major problem is seen during and after calculations.

## 5 Two-dimensional cavity flow

The scheme is extended to two-dimension, and applied to the cavity flow problems. No artificial dissipation is used. In the initial state, the fluid is at rest. The boundary condition is depicted in Figure 5. The pressure boundary condition is:

$$\nabla p \cdot \mathbf{n} = 0$$

(35)

Where $\mathbf{n}$ is the local normal to the wall. The governing equations are:

$$
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} &= \nu \Delta u \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} &= \nu \Delta v
\end{align*}
$$

(36)

The pressure is obtained by solving the following equation derived from eqns(36), using the conventional second-order central differencing and a line-by-line Gauss Seidel method.

$$
\Delta p_{n+1} = -2 \left[ \frac{\partial v^n}{\partial x} \frac{\partial u^n}{\partial y} - \frac{\partial u^n}{\partial x} \frac{\partial v^n}{\partial y} \right] \Delta t
$$

(37)

The velocity field is solved using the pressure obtained from eqn(37):

$$
\begin{align*}
\frac{\partial u_{i,j}}{\partial t} + \sum_{pqrs} (u_{ip,jq}u_{kq,hs} + v_{ip,jq}u_{kq,hs}) I_{pq} I_{rs} + \sum_r p_{i+r,j} = \nu \Delta u \\
\frac{\partial v_{i,j}}{\partial t} + \sum_{pqrs} (u_{ip,jq}v_{kq,hs} + v_{ip,jq}v_{kq,hs}) I_{pq} I_{rs} + \sum_r p_{i,j+r} = \nu \Delta v
\end{align*}
$$

(38)

Where the convection terms and pressure gradient terms are discretized, or projected onto space spanned by the two-dimensional scaling functions, as follows:

$$
\begin{align*}
\frac{\partial u_{i,j}}{\partial x} &= \sum_{n,m} \phi(x-i) \phi(y-j) \frac{\partial \phi(x,i)}{\partial x} (x,i) dx \right) dy \\
\frac{\partial v_{i,j}}{\partial y} &= \sum_{n,m} \phi(x-i) \phi(y-j) \frac{\partial \phi(x,i)}{\partial y} (x,i) dy \right) dx
\end{align*}
$$

(39)

$$
\begin{align*}
\frac{\partial p_{i,j}}{\partial x} &= \sum_k \phi(x-i) \phi(y-j) \frac{\partial \phi(x-k)}{\partial x} (x,i) dx \\
\frac{\partial p_{i,j}}{\partial y} &= \sum_k \phi(x-i) \phi(y-j) \frac{\partial \phi(x-k)}{\partial y} (x,i) dy
\end{align*}
$$

(40)
Obtained vorticity contours are shown in figure 6 to 8. The contour lines are drawn from 5 to -5, and the interval is divided in 30. The cases with Reynolds number 1000 and 10000 show characteristic structures of vorticities obtained by Shrieber and Keller [3] very well. At Reynolds number 2000, as seen in figure 8, the circular structure is distorted and the stationary state may no longer exist.

The velocity profiles along x=0.5 obtained with the schemes and those obtained by Shrieber and Keller [3] are compared in figures 9 and 10. In figure 9, the two are almost identical. At Reynolds number 10000, in figure 10, the two are almost identical near the walls, however the two profile differs in between. To explain the discrepancy, let us introduce the following variable:

\[
Y^n = \log_{10} \left( \sum |u_{ij}^{n+1} - u_{ij}^n| / \sum |u_{ij}^n| \right)
\]  

(41)

If \{u_y\} does not progress anywhere, \(Y^n\) goes to the negative infinity. In figure 11, however, \(Y\), as a function of time, is still fluctuating near \(t=120\) and slow oscillation is also observed. Shrieber and Keller assumed the existence of the stationary state and solved the stream function not progressively, but statically. The solution may actually be a possible solution. If the velocity field obtained by the
current scheme is averaged over some period, the two profiles may match.

6 Turbulent Viscosity

6.1 Motivation

Riemann problems with discontinuity are solved so far successfully with the dissipation model. The mathematical ground is the moment theorem. However if we did not know the theorem, the physical interpretation alone would derive the scheme. The physical interpretation described in the paper is something more than the so-called upwinding, although to the correctness of the interpretation is still to be examined. In the previous chapter, the artificial dissipation terms are not used. From the physical interpretation, the artificial dissipation terms could help simulate turbulent mixing with relatively coarse mesh.

6.2 Construction of scheme and results

If the first order Euler discretization in time is applied to eqns(36), we obtain equations for $u^{n+1}(x)$ and $v^{n+1}(x)$. To extract the velocity fluctuation from them, wavelets in two-dimension are needed. In two-dimension, one scaling function has corresponding three wavelets:

$$\langle x | \phi_n \psi_j \rangle = \phi_n(x) \psi_j(y), \langle x | \psi_n \phi_j \rangle = \psi_n(x) \phi_j(y), \langle x | \psi_n \psi_j \rangle = \psi_n(x) \psi_j(y)$$

(42)

For the sake of simplicity, the third wavelet is not used. By using the former two wavelet, for velocity fluctuation field are obtained:
From the wavelet coefficients, the random particle energy is calculated:

$$e^{n+1}(x) = e^n(x) + \sum \left( \left( \frac{\text{energy turbulent viscosity term for } U}{\|u_{1ij}\|^2 + \|u_{2ij}\|^2 + \|v_{1ij}\|^2 + \|v_{2ij}\|^2} \right) \psi_i(x) \phi_j(y) \right)$$  \hspace{1cm} (44)

From the energy turbulent viscosity term for $u$ becomes as follows:

$$\frac{\partial}{\partial x} \left( \lambda \sqrt{e^n} \frac{\partial u^n}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda \sqrt{e^n} \frac{\partial u^n}{\partial y} \right)$$  \hspace{1cm} (45)

In eqn(45), replacing $u$ with $v$ gives the turbulent viscosity term for $v$.

During the simulation of cavity flow at Reynolds number 10000, a passive scalar $f$ is injected. $f$ share the same simple boundary condition with $u$. Figures 12 to 14 show contours of $f$. The contour lines are drawn from 0.1 to 0.9 and the interval is divided in 40. The calculation produced figure 12 has used 201x201 scaling functions, hence it can well be a direct simulation. It is seen that the passive scalar spreads over almost entire region. The figure 13 shows the contours of $f$ calculated with 101x101 scaling functions, which is coarse for the Reynolds number, and the scalar stays upper half of the cavity. The figure 14 shows contours calculated with 101x101 functions too. However the turbulent viscosity terms have been added. The scalar spreads over almost entire region like figure 12, although figures 12 and 14 do not look like the same.

7 Concluding remarks

(1) By using wavelet decomposition of nonlinear convection terms, a new accurate and stable class of scheme is constructed. It is remarkable that the scheme captures steep shocks appearing in Riemann problems very clearly.

(2) The scheme is extended to two-dimension and solved cavity flow problems. The comparison of the results with those of Shrieber and Keller proves the validity and accuracy of the scheme.

(3) Turbulent mixing is simulated by a wavelet decomposition method. The turbulent mixing model created in the paper has produced decent results. Since the model is quite simple, intensive investigations are desired.

(4) The artificial dissipations models certainly has a mathematical justification, hence the stability analysis, which are no discussed here, improvement of performances will almost purely mathematically done. One characteristic of the scheme is that it has a physical interpretation that comes from an analogy with statistical physics. The improvement of capability to simulate physical phenomena will be achieved by sophistication of the current interpretation.
Figure 12: Passive Scalar Contours with No Turbulent Viscosity Terms
(Re=10000, t=40, 201x201)

Figure 13: Passive Scalar Contours with No Turbulent Viscosity Terms
(Re=10000, t=40, 101x101)

Figure 14: Passive Scalar Contours with Turbulent Viscosity Terms
(Re=10000, t=40, 101x101)

References