Drift forces on a floating or submerged spherical structure by multipole expansion method

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Abstract

A theoretical investigation about the drift forces on a spherical structure in waves of a finite depth ocean has been presented in this paper. We have used the linearised potential theory to obtain the velocity potential by use of the multipole expansion methods with the help of a Green function. The solution contains terms of an infinite series of associated Legendre functions with unknown coefficients. The series expression for the second order mean forces (drift forces), is provided by integrating the fluid pressure over the body surface. When the body is surface-piercing, in addition to the hydrodynamic drift forces, there exists the waterline drift forces. We are presently working on this problem to obtain analytical and numerical calculations. This study should be treated as a preliminary step forward for more complex investigation of offshore engineering problems.
1 Introduction

Wave induced drift forces acting on a submerged body in water have significant implications in a variety of practical applications. Many previous researchers including Newman [1], Pinkster and Van Ooortmerssen [2] and Wu et al [3] considered numerical and analytical approaches to calculate the drift forces based on the far-field and near field methods. Havelock [4], Hulme [5] and Gray [6] have done considerable research to obtain the analytical and numerical solutions of a moving body in ocean of finite and infinite depth. Wang [7] considered the problem of linear forces acting on a submerged sphere. Wu and Eatock-Taylor [8] considered a submerged spheroid and they solved the linear problem analytically. The radiation and diffraction of a submerged sphere were considered by Linton [9]. Thorne [10] was the first to demonstrate the method of multipole expansion to obtain the velocity potential for this linear diffraction problem.

In this paper we have demonstrated a solid mathematical formulation and solution techniques for the waterline and hydrodynamic drift forces acting on a floating spherical structure in waves. Analytical expressions have been found in a series form in terms of associated Legendre functions. Numerical simulation has been considered also. Detailed calculations will not be shown here. However, they can be found in the work of Rahman [11].

2 Mathematical Formulation

We assume that the fluid is homogeneous, inviscid and incompressible and the fluid motion is irrotational. A surface wave of a frequency $\sigma$ and of a small amplitude $A$ incident on a sphere of the radius $a$ submerged in water of a finite depth $d$ is considered. The wave period is $T = 2\pi/\sigma$.

We consider a right-handed Cartesian coordinate system $(x,y,z)$, in which the $(x,y)$-plane coincides with the free surface and the $z$-axis is taken vertically downwards from the free surface level. The geometric centre of the sphere is located at the point $(0,0,h)$, and $H = d - h$. Also, we consider the spherical coordinate system $(r,\theta,\psi)$ with the origin at the geometric centre of
the sphere. The relationship between these two sets of coordinate systems are as follows: \( z - h = r \cos \theta, \) \( x = r \sin \theta \cos \psi \) and \( y = r \sin \theta \sin \psi. \)

The fluid motion can be described by introducing a velocity potential \( \Phi(r, \theta, \psi, t) \)

\[
\Phi(r, \theta, \psi, t) = \text{Re} \left[ \phi(r, \theta, \psi) e^{-i\sigma t} \right],
\]

where \( \phi(r, \theta, \psi) \) is a complex potential.

The time-independent potential \( \phi(r, \theta, \psi) \) can be decomposed as \( \phi = \phi_I + \phi_D, \) where \( \phi_I \) is the incident potential, and \( \phi_D \) is the diffraction potential.

Thus we have to solve the following boundary value problem

\[
\nabla^2 \phi = 0, \tag{2}
\]

\[
\frac{\partial \phi}{\partial z} + K \phi = 0, \quad z = 0, \tag{3}
\]

\[
\frac{\partial \phi}{\partial z} = 0, \quad z = d, \tag{4}
\]

\[
\lim_{R \to \infty} \sqrt{R} \left( \frac{\partial}{\partial R} - ik_0 \right) \phi = 0, \tag{5}
\]

\[
\frac{\partial \phi_D}{\partial r} = - \frac{\partial \phi_I}{\partial r}, \quad r = a. \tag{6}
\]

where \( K = \frac{a^2}{a}, \) \( R = \sqrt{x^2 + y^2}, \) and \( k_0 \) is the finite depth wavenumber defined by \( k_0 \sinh k_0 d - K \cosh k_0 d = 0. \)

3 Incident potential

The incoming wave of the amplitude \( A \) and frequency \( \sigma, \) propagating in the positive \( x \)-direction, is described by the equation

\[
\phi_I = \frac{A \sigma \cosh k_0 (z - d)}{K \cosh k_0 d} e^{i k_0 R \cos \psi}, \tag{7}
\]

which can be expanded into a series of the associated Legendre functions \( P_n^m(x) \) as
\[ \phi_I(r, \theta, \psi) = \sum_{m=0}^{\infty} \phi_m^I(r, \theta) \cos m\psi, \]  

where

\[ \phi_m^I(r, \theta) = \frac{A_\sigma}{K} \epsilon_m i^m \sum_{s=0}^{\infty} \chi_s \frac{(k_0 r)^{s+m}}{(s + 2m)!} P_{s+m}(\cos \theta) \]  

with \( \epsilon_0 = 1 \) and \( \epsilon_m = 2 \) for \( m \geq 1 \) and

\[ \chi_s = \frac{(-1)^s e^{k_0(d-h)}}{2 \cosh k_0 d} \left\{ \begin{array}{ll}
\frac{\cosh k_0(d-h)}{\cosh k_0 d}, & s = 0, 2, 4, \\
\frac{\sinh k_0(d-h)}{\cosh k_0 d}, & s = 1, 3, 5, \ldots .
\end{array} \right. \]

The term \( P_n^m(\cos \theta) \) is defined as the associated Legendre function given by \( P_n^m(\cos \theta) = \sin^n \theta \frac{d^m P_n(x)}{dx^m} \big|_{x=\cos \theta} \) using the Legendre polynomial \( P_n(x) \) (see Abramowitz and Stegun [12]).

4 Diffraction potential

The diffraction velocity potential \( \phi_D \) can be expressed as

\[ \phi_D(r, \theta, \psi) = \sum_{m=0}^{\infty} \phi_m^D(r, \theta) \cos m\psi, \]

where \( \phi_m^D(r, \theta) \) is

\[ \phi_m^D(r, \theta) = \sum_{n=m}^{\infty} a^{n+2} A_n^m G_n^m(r, \theta). \]

Here \( A_n^m \) are the unknown complex coefficients to be found from a system of complex matrix equations, and

\[ G_n^m = \frac{P_n^m(\cos \theta)}{r_n^{n+1}} + \frac{P_n^m(\cos \alpha)}{r_1^{n+1}} + \frac{1}{(n-m)!} \\
\times \int_0^\infty \frac{(K + k)[e^{-k(d+H)} + (-1)^{n+m} e^{-kh}]}{(k \sinh kd - K \cosh kd)} \\
k^n \cosh k(z - d) J_m(kR) dk \]

with \( r_1 = \sqrt{R^2 + (d + H - z)^2} \) and \( \tan \alpha = \frac{R}{d + H - z} \). The improper line integral in eqn (13) has a singular point at \( k = k_0 \).
5 Total potential at the structure surface

We have got the expression for the total potential at the structure surface in the form

$$\phi_{ID}(\theta, \psi) = S(\theta, \psi) + \gamma, \quad (14)$$

where

$$S(\theta, \psi) = a \sum_{m=0}^{\infty} \sum_{n=\Delta(m)}^{\infty} \frac{2n+1}{n} A_n^m P_n^m (\cos \theta) \cos m\psi, \quad (15)$$

$$\gamma = \frac{A\sigma \cosh k_0(d - h)}{K \cosh k_0 d} + \sum_{n=1}^{\infty} A_n^0 a^{n+2} (B_{n0}^0 + C_{n0}^0), \quad (16)$$

and

$$\Delta(m) = \begin{cases} m, & m \neq 0 \\ 1, & m = 0 \end{cases} \quad (17)$$

It was found that $\gamma$ in eqn (16) can be a dominating term specially for small $K$. Previous researchers did not explicitly express this term but embedded in the general form of the solution.

6 Drift forces on the sphere

The hydrodynamic fluid pressure is given by Bernoulli’s equation

$$P = -\rho \left\{ \frac{\partial \Phi}{\partial t} + gz + \frac{1}{2} |\nabla \Phi|^2 \right\} \quad (18)$$

The first term on the right is called the transient pressure, the second term is the hydrostatic pressure and the third term is called the hydrodynamic dynamic pressure. The first order forces can be obtained from the first term. However, for the complete second order forces we need to consider all three terms on the right. It is worth mentioning that the hydrostatic term contributes to the second order forces only. The drift forces are obtained from the mean second order forces due to waterline forces (hydrostatic term) and the hydrodynamic forces (the velocity squared term).

Focusing on the first order forces and the drift forces, the wave induced pressures may be written as
\[ P = Re\{p_1 e^{-i\omega t}\} + \tilde{p}_2w + \tilde{p}_{2d} \]

where \( p_1 \) is the amplitude of linear pressure, and the second and third terms are respectively the time independent waterline and hydrodynamic pressures.

The waterline pressure at \( z = \eta \) is given by \( p_w = -\rho g \eta \), where \( \eta = -\frac{1}{g t} \). Thus the total waterline pressure at every point along the arclength around the sphere is given by

\[ p_{2w} = -\int_0^\eta \rho g zdz = -\frac{\rho}{2g} \left\{ \frac{\partial \Phi}{\partial t} \right\}^2 \quad (20) \]

Hence, the second order mean waterline pressure at every point along the arclength is given by

\[ \tilde{p}_{2w} = \frac{\rho \sigma^2}{4g} |\phi_{ID}|^2 \quad (21) \]

In a similar manner, the second order mean pressure can be written as

\[ \tilde{p}_{2d} = -\frac{\rho}{4} \nabla \phi_{ID} \cdot \nabla \phi_{ID}^* \quad (22) \]

where \( \phi_{ID} = \phi_I + \phi_D \) and the symbol * denotes the complex conjugate. The time-independent second-order pressure can be obtained in the form

\[ \tilde{p}_{2d}(\theta, \psi) = -\frac{\rho}{4} \left( |V_\theta(\theta, \psi)|^2 + |V_\psi(\theta, \psi)|^2 \right) , \quad (23) \]

where \( V_\theta = \frac{1}{r} \frac{\partial \phi_{ID}}{\partial \theta} \) and \( V_\psi = \frac{1}{r \sin \theta} \frac{\partial \phi_{ID}}{\partial \psi} \) are

\[ V_\theta(\theta, \psi) = \cot \theta \sum_{m=0}^\infty \sum_{n=\Delta(m)}^\infty (2n + 1) A_n^m P_n^m(\cos \theta) \cos m\psi \]

\[ - \csc \theta \sum_{m=0}^\infty \sum_{n=\Delta(m)}^\infty (2n + 1)(n + m) \frac{A_n^m P_n^m(\cos \theta) \cos m\psi}{n} , \quad (24) \]

\[ V_\psi(\theta, \psi) = -\csc \theta \sum_{m=1}^\infty \sum_{n=m}^\infty (2n + 1)m \frac{A_n^m P_n^m(\cos \theta) \sin m\psi}{n} \]

\[ A_n^m P_n^m(\cos \theta) \sin m\psi. \quad (25) \]
In evaluating these expressions we have used the associated Legendre functions identity \((x^2-1)\frac{dP_m^n(x)}{dx} = n x P_m^n(x) - (m+n)P_{m-1}^n(x)\) where \(x = \cos \theta\). Here it is to be noted that the radial velocity component \(V_r = \frac{\partial \phi_ID}{\partial r}\) at \(r = a\) is always zero. This is the body surface boundary condition.

The forces acting on the sphere can be calculated by integrating the fluid pressure over the body surface \(S_b\)

\[ F_j = -\int \int_{S_b} P n_j dS \]  \hspace{1cm} (26)

where the term \(n_j\) represent the components of the normal out of the body surface which can be expressed as \(n_j = P_1^j(\cos \theta) \cos j\phi\) with indices \(j = 0\) and \(j = 1\) corresponding to heave and surge motions respectively.

Drift forces usually defined as the mean second order forces due entirely to the first order velocity potential. When the second-order time dependent forces are ignored, the total forces can be written as

\[ F_j = Re[f_je^{-i\omega t}] + \bar{f}_j \]  \hspace{1cm} (27)

where \(\bar{f}_j = \bar{f}_{jw} + \bar{f}_{jd}\) contains both the waterline drift forces and hydrodynamic drift forces.

Substituting eqn (18) into eqn (26), the first order exciting forces and the second order drift forces are respectively given by

\[ f_j = i\rho \sigma \int \int_{S_b} (\phi_ID)n_j dS \]  \hspace{1cm} (28)

and the waterline drift forces are

\[ \bar{f}_{jw} = \frac{\rho \sigma^2}{4g} \int_w |\phi_ID|^2 n_j dS \]  \hspace{1cm} (29)

also the hydrodynamic drift forces are

\[ \bar{f}_{jd} = \frac{\rho}{4} \int \int_{S_b} |\nabla \phi_ID|^2 n_j dS \]  \hspace{1cm} (30)

The explicit expression and their numerical solutions can be found in Rahman [11] and will not be shown here because of the space limitation.
7 Discussion and conclusion

A theoretical study is made in this paper to evaluate the drift forces on a floating sphere in a finite depth ocean. It has been found that when the body is surface piercing there are two important drift forces will act on the body surface; one due to the hydrodynamic forces and the other due to the hydrostatic forces. These forces are usually determined through the first order velocity potential, although they contribute to the total second-order forces acting on the body. Lighthill [13] demonstrated in his pioneering work presented at the BOSS Conference held at Imperial College in 1979 that in general there are three second order forces act on a body surface; one due to the second order potential, second one due to fluid velocity and the third one due to the hydrostatic pressure. These forces, of course, can be mathematically derived from the Bernoulli equation. This paper considers only the drift forces which are induced due to first order velocity potential only. To solve the complete problem we need to obtain some expression for the second order potential or some relationship between the second order potential and the first order potential through the divergence theorem of vector calculus. That was exactly what Lighthill [13] has demonstrated in his lecture at Imperial College. In this paper we avoid any rigorous numerical calculation of drift forces. In our future work we will include extensive graphical simulations of these forces.

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References


