An explicit WET-Type model for the turbulent scalar fluxes

M. Rokni & B. Sundén

Division of Heat Transfer, Lund Institute of Technology, Box 118, 221 00 Lund, Sweden

Abstract

A new explicit WET model for the turbulent heat fluxes is derived and developed from the transport equation of the scalar passive flux. In order to study the performance of the model itself, all other turbulent quantities are taken from a DNS channel flow data-base and thus the error source has been minimized. This anisotropic eddy diffusivity model can be solved with a two-equation model for the temperature field based on the temperature variance and its dissipation rate. The model can theoretically be used for a range of Prandtl numbers or a range of different fluids. The results are compared with the DNS channel flow and fairly good agreement is achieved.

1 Introduction

It is known that the assumption of a constant turbulent Prandtl number is valid only for fluids with molecular Prandtl number of approximately unity. It is also known that the turbulent Prandtl number is not constant in the near wall region and increases when a solid wall is approached. In addition, the conventional simple eddy diffusivity model (SED) may not be adequate for non-adiabatic boundary conditions or fluids with very low (or very high) molecular Prandtl numbers. Probably, the major disadvantage of the SED model is that the model is incapable of predicting streamwise scalar flux. A better representative algebraic expression for turbulent scalar fluxes is based on the generalized gradient diffusion hypothesis (GGDH) presented by Daly and Harlow [1]. However, the crucial problem of the GGDH model is that this model gives an extreme under-prediction of the streamwise scalar flux, \( \bar{u} \phi \), while it predicts the wall-normal direction, \( \bar{v} \phi \), fairly well for moderate Prandtl numbers, even in simple wall
shear flows. To improve the model performance many advanced algebraic models for the scalar fluxes have been presented, among them Yoshizawa [2], So and Sommer [3], Abe et al. [4] and Wikström et al. [5] can be named. It will soon be shown that many of such models are entirely dependent on the expression of the turbulent Reynolds stress used in the respective study and may not perform satisfactory if other more advanced Reynolds stress modeling are used. For example, Abe et al. [4] used a simple eddy viscosity model for Reynolds stress modeling which has shown to be inadequate in more three-dimensional duct flows. So and Sommer [3] used a full Reynolds stress modeling (RSM) for the Reynolds stresses which is extremely time-consuming and may perform instable in more complex geometries. This would in turn imply that there still exists interest in developing a model which is less dependent on the kind of Reynolds stress expression, but instead can be used with more appropriate turbulent Reynolds stress models such as explicit algebraic stress models or anisotropic eddy viscosity models.

Simple eddy diffusivity models with non-constant turbulent Prandtl number in the level of a zero-equation model (Rokni and Sundén, [6]) or a two-equation model of the temperature variance and its dissipation-rate (So and Sommer, [7]) have been used in previous studies. However as pointed out, these models still suffer from their inability to predict the streamwise scalar flux.

In the present study, an implicit expression for the scalar fluxes is derived from the transport equation of the scalar-flux by modeling the dissipation of the temperature variance and the pressure-gradient correlation together with the assumption of negligible convection and diffusion of the scalar flux in homogenous steady flows. An explicit expression is then derived by assuming that an explicit algebraic stress model for the turbulent Reynolds stress and assuming the scalar flux is proportional to a varying diffusivity times temperature gradients. Because some of the assumptions made are not totally general, the authors have named the new derived expression as an explicit WET (Wealth \(\propto\) Earning x Times, [8]) model which is also an anisotropic eddy diffusivity model. The proposed model in its final form is a function of a non-constant diffusivity, mean strain-rate, mean vorticity, the invariant coefficients, turbulent viscosity with dynamic \(C_\mu\) and temperature gradients.

This new anisotropic eddy diffusivity model can be solved with the two-equation model of temperature variance and its dissipation rate. It is capable to be applied for various wall thermal conditions and can be used for fluids with molecular Prandtl numbers within a wide range to cover most fluids. Moreover, using a simple eddy diffusivity concept in the turbulent heat fluxes (including the two-equation model of temperature variance and its dissipation rate) does not allow for a proper prediction of all the turbulent heat fluxes while a anisotropic heat transfer model may do.

### 2 The deficiency of SED and GGDH models

The SED model is based on the turbulent viscosity and the turbulent Prandtl number, and is defined as
\[
\overline{u_i \theta} = -\alpha_t \frac{\partial \Theta}{\partial x_i} = -\frac{v_t}{Pr_t} \frac{\partial \Theta}{\partial x_i}
\]
which is not capable to predict the streamwise scalar flux, \( \overline{u\theta} \), and extremely over-predicts the wall-normal scalar flux, \( \overline{v\theta} \), close to the solid wall, see Fig. 1. Such deficiencies may be avoided if the scalar fluxes are also proportional to the turbulent Reynolds stresses as e.g. proposed in the GGDH model of Daly and Harlow [1],
\[
\overline{u_i \theta} = -\alpha \left( \frac{k}{\varepsilon} \frac{u_i u_j}{\varepsilon} \frac{\partial \Theta}{\partial x_j} \right)
\]
or WET model of Launder [8],
\[
\overline{u_i \theta} = -\alpha \left( \frac{k}{\varepsilon} \left[ \frac{u_i u_j}{\varepsilon} \frac{\partial \Theta}{\partial x_j} + u_j \theta \frac{\partial U_i}{\partial x_j} \right] \right)
\]
However, this model predicts the wall-normal scalar flux very well, but it extremely under-predicts the streamwise scalar flux, \( \overline{u\theta} \), as shown in Fig. 1. In addition, such a model without further modifications cannot be used for fluids with very high or very low Prandtl numbers. As Launder [8] pointed out, the utility of this model depends on how accurately the component values of the Reynolds stress tensor are known. This has also been observed during the course of this study and will partly be demonstrated later.

In order to minimize the errors of all parameters which appear in both SED and GGDH models, the DNS results of Kasagi and Lida [9] is used, which is provided for a parallel plate channel flow with constant wall temperature and Reynolds number of 4560 based on the channel width, and with the molecular Prandtl number of 0.71. In other words, all the turbulent parameters such as kinetic energy, dissipation rate, Reynolds stress and temperature gradients are taken from these DNS data and, afterwards, the scalar fluxes are calculated. Finally, these calculated scalar fluxes are compared with those given in the same DNS data-base, and therefore the errors has been minimized which otherwise may appear by, e.g., modeling the Reynolds stresses and the dissipation rates. In eqns. (2) and (3) the constant \( C_0 \) is set to 0.3 and in eqn. (1) the \( Pr_t \) is set to 0.89 which are adopted widely among researchers. The turbulent viscosity in eqn. (1) is calculated from
\[
v_t = C_\mu \frac{k^2}{\varepsilon}
\]
with \( C_\mu = 0.09 \). Including a damping function or not into the turbulent viscosity is questionable. Because both the kinetic energy and its dissipation-rate are taken from the DNS data, one may argue that there is no need for any damping function. On the other hand, the value 0.09 for \( C_\mu \) is only valid at the high Reynolds numbers and in order to consider its variation close to a solid wall a damping function is needed. Therefore two version of the SED is included in Fig 1., one without the damping function in the turbulent viscosity (referred as SED
in the figure) and another one including a damping function in eqn. (4) (referred as in the SED 2). The damping function used here is from Abe et al. [4]

\[ f_\mu = \left(1 - \exp\left(-\frac{y^*}{14}\right)\right)^2 \left(1 + \frac{5}{\text{Re}_1^{0.75}} \exp\left[-\left(\frac{\text{Re}_1}{200}\right)^2\right]\right) \]  

(5)

where \( y^* = \frac{u'_e}{v} = (\sqrt{\text{Re}_k})^{0.25} \frac{\eta}{v} \) and \( \text{Re}_1 = \frac{\rho k^2}{\mu e} \).  

(6)

It should be noted that the over-prediction of the wall-normal scalar flux by SED model (in the near wall region) can be somewhat overcome if the damping function is included in the turbulent viscosity (SED 2 in the figure). This implies that the argument about decreasing \( C_\mu \) close to a solid wall is true. Another model which uses a non-constant \( \text{Pr}_k \) via the turbulent diffusivity as discussed in Rokni and Sundén [6], hereafter called as MSED model (Modified SED), is also included for comparison. However, the MSED model behaves similar to the SED with a damped \( C_\mu \) in capturing the wall-normal scalar flux. All the SED-based models under-predict this component in the region away from the wall while both the GGDH and WET models capture this scalar component satisfactorily in comparison with the DNS data.

Neither the traditional SED models nor the modified one are capable in capturing the streamwise scalar flux (zero prediction), as shown in Fig. 1. The GGDH and WET models provide a non-zero prediction for the streamwise scalar flux, although GGDH extremely under-predicts the peak value. The WET model provides a more than twice larger peak than the GGDH model and therefore this model (or any model with similar expression for the scalar fluxes as the WET model) may be regarded as a model that has a high potential for improvement. However, the WET-type models are implicit in nature and an explicit form is desired.
Fig. 1 clearly demonstrates that there exists a need to provide a model which overcome the deficiencies of the commonly used SED and GGDH models.

3 A new explicit WET-type model for scalar fluxes

The development of an anisotropic algebraic eddy diffusivity model starts from the governing equation for the turbulent scalar fluxes written as (if the buoyancy effect is neglected):

$$\frac{D u_i \theta}{Dt} - D_{\theta i} = P_{\theta i} + \Pi_{\theta i} - \varepsilon_{\theta i}$$

(7)

where $D_{\theta i}$ is the molecular and turbulent diffusion of $u_i \theta$, $\Pi_{\theta i}$ is the pressure scalar-gradient correlation, $\varepsilon_{\theta i}$ is the destruction rate-tensor and the production term is given by

$$P_{\theta i} = P + P_\theta = -u_i u_j \frac{\partial \Theta}{\partial x_j} - u_j \frac{\partial u_i}{\partial x_j}$$

(8)

Assuming the IPM theory (Isotropization of Production Model), the dissipation rate $\varepsilon_{\theta i}$ and the pressure scalar-gradient correlation $\Pi_{\theta i}$ are modeled as

$$\Pi_{\theta i} - \varepsilon_{\theta i} = -C_{\theta i} \frac{1}{\tau} u_i \theta + C_{\theta 2} u_j \theta \frac{\partial u_i}{\partial x_j} + C_{\theta 3} u_j \theta \frac{\partial u_j}{\partial x_i} + C_{\theta 4} u_i u_j \frac{\partial \Theta}{\partial x_j}$$

(9)

which is the most general linear form with conserving the super-position principle of passive scalars and has been widely used among researchers, see e.g. Launder [10], Shabany and Durbin [11]. $\tau$ is an appropriate time-scale and $C_{\theta i}$ through $C_{\theta 4}$ are the model constants. The time-scale $\tau$ might be a combination of both velocity time-scale ($k/\varepsilon$) and scalar time-scale ($\Theta/\varepsilon_\theta$), and then it can be regarded as a mixed time-scale and will be discussed later. Using assumption of similarity between the transport of $u_i \theta$ and the transport of k- $\theta$ (Pop, [12]), the left hand side of eqn. (7) can be approximated as

$$\frac{D u_i \theta}{Dt} - D_{\theta i} \approx \frac{u_i \theta}{k} (P - \varepsilon) + \frac{u_i \theta}{\Theta} (2P_\theta - 2\varepsilon_\theta)$$

(10)

All the notations are the standard ones, see e.g. Pop [12]. Assuming that local equilibrium prevails for both velocity and scalar fields then the right-hand-side of eqn. (9) becomes zero which in turn simplifies eqn. (7) through eqns. (8-10) to

$$-u_i \theta = \frac{\tau}{C_{\theta i}} \left[ (1 - C_{\theta 4}) u_i u_j \frac{\partial \Theta}{\partial x_j} + (1 - C_{\theta 2}) u_j \theta \frac{\partial u_i}{\partial x_j} - C_{\theta 3} u_j \theta \frac{\partial u_j}{\partial x_i} \right]$$

(11)

which is a classical approach and in fact can be identified as the WET model of Launder [8] if $C_{\theta 2}$ through $C_{\theta 4}$ are set to zero. This is an implicit expression and singularities/instabilities may occur for certain behavior of $u_i u_j$ and the mean gradient of $U_i$, or during the iteration process. Therefore a different approach is needed to avoid such situations. In addition, it is quite common that both $C_{\theta 3}$ and
C_{t1} are assumed to be very small [5] and can be set to zero, which is also assumed in this study.

The original definition of the time-scale \( \tau \) appearing in the above equations is \( k/\varepsilon \). However, two other different mixed time-scales have been used by previous researchers; \( \tau = \left( k/\varepsilon \right) \left( k/\varepsilon_0 \right)^{0.5} \) and \( \tau = \left( \varepsilon/k \right) \left( k/\varepsilon_0 \right)^{2} \). None of these two time-scales has been shown to be clearly superior. In this study, the former definition of the time-scale is used. It represents an arithmetical approximation of both the dynamic and scalar time-scales.

In order to derive an explicit version of eqn. (11), the Reynolds stresses and the scalar fluxes must be modeled. Similar to the GGDH model this equation requires a second order accuracy for the Reynolds stresses and the linear-eddy viscosity model cannot be used, which also was discussed by Launder [8]. The low-Reynolds number Explicit Algebraic Stress Model (EASM) of Rokni [13] (the low Reynolds version of Gatski and Speziale [14]) is taken as a low-order approximation to the full ten-terms basis for \( \bar{u}_i \bar{u}_j \) as

\[
\bar{u}_i \bar{u}_j = \frac{2}{3} k \delta_{ij} - 2 \nu_{r1}^* S_{ij} - 2 \nu_{r2}^* \frac{k}{\varepsilon} \left( S_{ik} W_{kj} + S_{jk} W_{ki} \right) + 4 \nu_{r3}^* \frac{k}{\varepsilon} \left( S_{ik} S_{kj} - S_{mn} S_{mn} \delta_{ij} \right)
\]

(12)

where the strain and rotation rates are

\[
S_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \quad W_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \right).
\]

(13)

It has been shown by many researchers (see, e.g., [4,7,10]) that a simple gradient transport approximation for \( \bar{u}_i \bar{\theta} \) can recover the wall-normal scalar flux properly. Such an approximation is also assumed in the right hand side of eqn. (11). This approximation is not the most proper but is used temporarily to find an explicit formulation for the algebraic scalar fluxes. Therefore, the formulation, shown below, is here named as Explicit WET model (EWET) rather than an explicit algebraic scalar flux model. Inserting eqns. (12) and (13) into eqn. (11) with the latter approximation provides the following expression for the scalar fluxes as

\[
- \bar{u}_i \bar{\theta} = \frac{2 \tau k}{3 C_{t1}} \frac{\partial \bar{\theta}}{\partial x_i}
- \frac{\tau}{C_{t1}} \left[ 2 \nu_{r1}^* + (1 - C_{t2} - C_{t3}) \alpha_t \right] S_{ij} + (1 - C_{t2} + C_{t3}) \alpha_t W_{ij} \frac{\partial \bar{\theta}}{\partial x_j}
- \frac{2 \tau}{C_{t1}} \nu_{r2}^* \frac{k}{\varepsilon} \left( S_{mn} W_{mj} + S_{jm} W_{mi} \right) \frac{\partial \bar{\theta}}{\partial x_j}
+ \frac{4 \tau}{C_{t1}} \nu_{r3}^* \frac{k}{\varepsilon} \left( S_{mn} S_{mj} - \frac{1}{3} S_{mn} S_{mn} \delta_{ij} \right) \frac{\partial \bar{\theta}}{\partial x_j}
\]

(14)
where

\[ v_{\tau,i}^* = \frac{f_\mu}{\varepsilon} C_{\mu,i}^* \frac{k^2}{\varepsilon} \quad i = 1,2,3 \]  
(15)

\[ C_{\mu,1}^* = \alpha_1 \left\{ \frac{3(l + \eta^2) + 0.2(\eta^6 + \xi^6)}{\left[ 3 + \eta^2 + 6\xi^2 \eta^2 + 6\xi^2 + \eta^6 + \xi^6 \right]} \right\} \]  
(16)

\[ C_{\mu,i}^* = \alpha_1 \alpha_i \left\{ \frac{3(l + \eta^2)}{\left[ 3 + \eta^2 + 6\xi^2 \eta^2 + 6\xi^2 + \eta^6 + \xi^6 \right]} \right\} \quad i = 2,3 \]  
(17)

\[ \alpha_1 = 0.114, \quad \alpha_2 = 0.187, \quad \alpha_3 = 0.088 \]  
(18)

\[ \eta = \frac{f_\mu}{\varepsilon} \alpha_3 \frac{k}{\varepsilon} (S_{ij} S_{ij})^{0.5}, \quad \xi = \frac{f_\mu}{\varepsilon} \alpha_2 \frac{k}{\varepsilon} (W_{ij} W_{ij})^{0.5} \]  
(19)

A comparison of \( \vec{\nabla} \vec{\theta} \) from eqn. (14) and \( u_{ij} \theta = -\alpha_t \partial \theta / \partial x_i \) in the near wall region suggests that \( \tau_k/(3C_{\theta 1}) \) must be the turbulent diffusivity \( \alpha_t \). The scalar flux model takes thus its final form as

\[- u_{ij} \theta = \alpha \frac{\partial \theta}{\partial x_i} \]

\[- \frac{\tau}{C_{\theta 1}} \left\{ \frac{2v_{\tau,1}^* + (1 - C_{\theta 2} - C_{\theta 3}) \alpha_t S_{ij} + (1 - C_{\theta 2} + C_{\theta 3}) \alpha_t W_{ij}}{\partial x_j} \right\} \]

\[- \frac{2\tau}{C_{\theta 1}} v_{\tau,2}^* \frac{k}{\varepsilon} \left( S_{mn} W_{mj} + S_{jm} W_{mj} \right) \frac{\partial \theta}{\partial x_j} \]

\[ + \frac{4\tau}{C_{\theta 1}} v_{\tau,3}^* \frac{k}{\varepsilon} \left( S_{mn} S_{mj} - \frac{1}{3} S_{mn} S_{mj} \delta_{ij} \right) \frac{\partial \theta}{\partial x_j} \]  
(20)

In this study the turbulent diffusivity \( \alpha_t \) is defined as

\[ \alpha_t = C_\lambda \left\{ \frac{k^2}{\varepsilon} + 3k^{0.5} \left( \frac{v^3}{\varepsilon} \right)^{0.25} \frac{1}{\Pr} f_d \left( 1 - e^{-\left( \frac{y^*}{34} \right)} \right) \right\} \left( 1 - e^{-\frac{\sqrt{Pr} y^*}{34}} \right) \]  
(21)

which was proposed based on the studies of Rokni and Sundén [6] and Abe et al. [4]. Of course, other definitions such as \( \alpha_t = C_\lambda f_d k \tau \) can also be used instead of eqn. (21). However, eqn. (21) can be used for any wall-boundary condition as was carefully discussed by Abe et al. [4] and therefore it is used in this study. The constants and the damping function in eqns. (20) and (21) are set to:

\[ C_{\theta 1} = 3.33, \quad C_{\theta 2} = 0.4, \quad C_\lambda = 0.35, \quad f_d = \exp \left\{- \left( \frac{Re_i}{200} \right)^2 \right\} \]  
(22)

As has been pointed out earlier, that eqn. (11) can be identified as the WET model of Launder [8] if \( C_{\theta 2} \) through \( C_{\theta 4} \) are set to zero and the constant in front of the parenthesis is set as the \( C_{\theta} \) in the GGDH model. Therefore \( C_{\theta 1} \) is set to 3.33 which is the reciprocal of 0.3. The constant \( C_{\theta 2} \) is taken from the study of
So and Sommer [3] without further modification. The constant $C_{03}$ is set to zero as was discussed earlier. The reader should thus be aware that these constants are not calibrated for the proposed model and are taken from previous studies.

In eqn. (21), $y^*$ is the non-dimensional normal distance to the wall based on the Kolmogorov velocity scale rather than the friction velocity. Using the Kolmogorov velocity scale allows for accounting the near wall effect in both attached and detached flows without singularities. It is defined in eqn. (6).

A comparison of the final form of presented EWET model, eqn. (20), with the SED model, eqn. (1), shows that the first term in eqn. (20) is in fact the SED part and then additional terms are included. These extra terms may be regarded as the anisotropic part of the model.

### 3 Results and discussions

In order to study the behavior of the scalar flux model itself, it is preferred to build up a frame based on the DNS data and only calculate the scalar fluxes themselves. Therefore, all other quantities such as velocities, turbulent Reynolds stresses, kinetic energy and its dissipation rate, temperature, temperature variance and its dissipation rate are taken from suitable/available DNS results. Such a DNS data base is presented and released by Kasagi and Lida (1999) for a parallel plate channel flow with constant wall temperature, $Re = 4560$ and turbulent Reynolds number ($Re_t$) as 150 with molecular Prandtl-number as 0.71. In Fig. 2 all the turbulent quantities are thus taken from the DNS data except scalar fluxes, and thereby the scalar fluxes are calculated by different models and compared with the same DNS data. This way is the most efficient way to assess the ability of a proposed model and minimize errors which otherwise may occur.

![Figure 2](image-url)  
*Figure 2. Prediction of the scalar flux components $\overline{u'v'}$ and $-\overline{v't}$ by different models in comparison with the DNS data. SED = Simple Eddy Diffusivity, SS = So and Sommer (1995), AKN = Abe et al. (1995).*
As can be seen from the figure, both SED and SS models are unable to predict the component $-\bar{v}\theta$ near the wall properly, while they are able to capture the DNS data far away from the solid wall. The inability of the SED model near the wall is due to the constant turbulent Prandtl number model. In addition, the AKN model behaves physically incorrect in a small region near the solid wall (the negative values), while its overall performance may be regarded better than the SED and SS models. The present model performs best, although it has some problem near the wall.

It can be noted that the SS and AKN models predict this component very well if they use the turbulence model proposed by the respective authors. This means that the errors appearing in the modeling of SS and AKN for other turbulent quantities than scalar fluxes compensated the errors of their scalar flux modeling and thus satisfactory results were provided. Furthermore, Fig. 2 is a demonstration how a model may perform if other turbulent Reynolds stress modeling than the proposed one, by the respective authors is used. It is known that some of these Reynolds stress models do not perform well in more complicated geometries. In other words, if such models are used with a more proper Reynolds stress tensor expression, then their performance in capturing the scalar flux components may not be satisfactory.

However, the presented model performs very well in predicting the scalar flux components when all turbulent quantities (except scalar fluxes) from DNS data are used. This would imply that if the presented model is applied without using DNS data and instead turbulence modeling is used for different quantities such as Reynolds stresses, temperature variance and its dissipation-rate and dissipation of kinetic energy, then the source of error lays in these equations, not in the presented scalar flux model.

As it is known, the most severe disadvantage of the SED model is that it gives a zero value for the component $\bar{u}\theta$. The SS model exaggerates this component near the wall while the AKN model under-predicts it. The presented model captures the peak satisfactorily and performs much better overall.

4 Conclusions

A new model was presented for the prediction of the turbulent scalar components based on some basic physical stand points, such as IPM theory, local equilibrium, linear pressure-scalar gradient, second-order definition for the Reynolds stresses and an arithmetic approximation between the dynamical and scalar time-scales. This new model is as anisotropic eddy diffusivity model and can also be regarded as a first step to a fully explicit algebraic scalar flux model.

This new model has minimum source of errors compared with other models studied, because it performs very well when other turbulent quantities are used from the DNS data.

The streamwise scalar flux is very sensitive to the expression of the Reynolds tensor and other turbulent quantities. This implies that if a scalar flux model is developed from improper turbulent quantities, then there may exist some errors
in the scalar flux model itself which compensates the errors of other turbulent quantities and finally performs only with the proposed turbulence models, but not with more elaborate turbulence modeling of Reynolds stresses, temperature variance and it dissipation-rate.

5 References